ELLIPTIC EQUATIONS IN WEIGHTED BESOV SPACES ON ASYMPTOTICALLY FLAT RIEMANNIAN MANIFOLDS

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ABSTRACT. This paper deals with the applications of weighted Besov spaces to elliptic equations on asymptotically flat Riemannian manifolds, and in particular to the solutions of Einstein's constraints equations. We establish existence theorems for the Hamiltonian an momentum constraints with constant mean curvature and with a background metric which satisfies very low regularity assumptions. These results extend the regularity results of Holst, Nagy and Tsogtgerel about the constraint equations on compact manifolds in the Besov space $B^s_{p,p}$ [18], to asymptotically flat manifolds. We also consider the Brill–Cantor criterion in the weighted Besov spaces. Our results improve the regularity assumptions on asymptotically flat manifolds [13, 23], as well as they enable us to construct the initial data for the Einstein–Euler system.

1. INTRODUCTION

Much attention has been devoted to solutions of the Einstein constraint equations in asymptotically flat space–times by means of weighted Sobolev spaces as an essential tool. These spaces are defined by the norm

$$\|u\|_{m,p,\delta} = \left(\sum_{|\alpha|\le m} \int \left|(1+|x|)^{\delta+|\alpha|}\partial^\alpha u\right|^p dx \right)^{\frac{1}{p}}, \tag{1.1}$$

and denoted by $W^p_{m,\delta}$.

Elliptic equations on $W^p_{m,\delta}$ spaces were first considered by Nirenberg and Walker in [25]. This paper led to numerous publications dealing with its applications to the solutions of Einstein constraint equations in asymptotically flat space–times. Some significant contributions include the papers of Bartnik [3], Cantor [8, 9], Choquet–Bruhat and Christodoulo [12] and Christodoulou and O'Murchadha [15]. Afterward the regularity assumptions were improved, by Maxwell in the vacuum case and with boundary conditions [23], and by Choquet–Bruhat, Isenberg and Pollack for the Einstein–scalar field gravitational constraint equations [13]. In both papers they obtained that the metric is locally in W^p_2 when $p > \frac{n}{2}$.

2010 *Mathematics Subject Classification.* Primary 35J91, 58J05; Secondary 35J61, 83F99.

Key words and phrases. Asymptotically flat Riemannian manifolds, weighted Besov spaces, elliptic equations, constraint equations, Brill–Cantor criterion, Einstein–Euler systems .

*Research supported ORT Braude College's Research Authority.

In the case $p = 2$, Maxwell constructed low regularity solutions of the vacuum Einstein constraint equations with a metric in the Bessel potential spaces H^s_{loc} with $s > \frac{n}{2}$ [24].

The extension of the $W^p_{m,\delta}$-spaces to the weighted Besov spaces of fractional order was carried out by Triebel [29] (see definition 2.1). We denote these spaces by $W^p_{s,\delta}$, where $s, \delta \in \mathbb{R}$ and $p \in [1, \infty)$. In this paper we are applying the weighted Besov spaces to the study of elliptic equations on asymptotically flat Riemannian manifolds. This enable us to improve regularity results. We also construct the initial data for the Einstein fields which are coupled to a perfect fluid, and whose density either has a compact support or falls off at infinity in an appropriate manner.

On compact manifolds one seeks solutions to the Einstein constraint equations in the unweighted Sobolev spaces. Choquet–Bruhat obtained solutions with metric in W^p_2 and for $p > \frac{n}{2}$ [11]. Later Maxwell improved the regularity in the Bessel potential spaces H^s for $s > \frac{n}{2}$ [22]. Holst Nagy and Tsogtgerel [18] study solutions of the Einstein constraints in the Besov space $B^s_{p,p}$ (see (2.1)). Among their results is the existence of weak solutions to the Hamiltonian and momentum constraints in the Besov space $B^s_{p,p}$ with $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $p \in (1, \infty)$. Thus their results cover [11] in the case $s = 2$ and [22] in the case $p = 2$.

The present paper extends the regularity results of Holst Nagy and Tsogtgerel on compact manifolds [18], to asymptotically flat Riemannian manifolds by using the weighted Besov spaces $W^p_{s,\delta}$ (see (2.1)). We establish existence results for the Hamiltonian constraint with constant mean curvature (CMC) and the momentum constraint under the conditions $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, n - 2 - \frac{n}{p})$ and for all $p \in (1, \infty)$. The Brill–Cantor condition suggests a criterion, similarly to the Yamabe number, under which a given metric in asymptotically flat manifold can be rescaled to yield a conformal metric with zero scalar curvature (see §5). For an enlightening discussion about this criterion see [17]. The equivalence between Brill–Cantor condition and a flat metric was solved for the integer order weighted Sobolev spaces and for $p > \frac{n}{2}$ in [13, 23]. We treat the Bill–Cantor condition in the weighted Besov spaces $W^p_{s,\delta}$ and establish its equivalent to a metric with zero scalar curvature for $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, n - 2 - \frac{n}{p})$ and all $p \in (1, \infty)$. To conclude, our results generalize [7, 24] to $p \in (1, \infty)$, and improve the regularity of [13, 23].

The outline of the paper is as follows. In the first part of Section 2 we sketch Triebel's construction of the weighted Besov spaces and state their main properties. In the second part we establish tools which are needed for PDE in these spaces, including embeddings, pointwise multiplication and Moser type estimates.

Section 3 is devoted to elliptic linear systems on asymptotically flat Riemannian manifolds. In the first subsection we establish *a priori* estimates for second order elliptic operators with coefficients in the $W^p_{s,\delta}$ spaces and show that these system are semi–Fredholm operators. This property has an essential role in the studying of the non–linear equations. The definition of asymptotically flat of the class $W^p_{s,\delta}$ is done in subsection 3.2. In subsection 3.3 we are studying of weak solutions that meet very low regularity requirements. This demands a special attention to the extension of L^2-bilinear forms to the product of $W^p_{s,\delta}$ with

its dual on the manifold. Here we follow the ideas of [18, 22] and establish the continuous extensions (Propositions 3.9 and 3.10). We then define weak solutions on the manifolds and derive a weak maximum principle.

In Section 4 we prove the existence and uniqueness theorem of a semi–linear equation, where the linear part is the Laplace–Beltrami operator of an asymptotically flat Riemannian manifold. The method of sub and super solution is the common method for these types of non–linearity, however, we shall implement Cantor's homotopy argument [9] in the weighted Besov spaces.

Section 5 discusses the Brill–Cantor criterion. We show that condition (5.1) is necessary and sufficient for the existence of a metric in $W^p_{s,\delta}$ and with zero scalar curvature, when $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, -2 - n + \frac{n}{p})$ and all $p \in (1, \infty)$. The main difficulty is the estimate of the Yamabe's functional (5.1) in terms of the relevant $W^p_{s,\delta}$-norms. Here we use a slightly advanced form of pointwise multiplication of three functions in $W^p_{s,\delta}$ (see Proposition 2.11). Finally, in Section 6 we are considering the construction of initial data for the Einstein–Euler system. In this system the equations for the gravitational fields are coupled to a perfect fluid and certain relations between the matter variables and the fluid variables must be fulfilled. In [7] the authors discussed this problem in details in the weighted Hilbert spaces $W^2_{s,\delta}$. Here we extend these results to the $W^p_{s,\delta}$ spaces.

Some notations: For $p \in (1, \infty)$, p' will stand for the dual index to p, that is $\frac{1}{p} + \frac{1}{p'} = 1$. The scaling of a distribution u with a positive number ε is denoted by u_ε. A Riemannian manifold is denoted by $\mathcal{M}$ and $g = g_{ab}$ is a metric on $\mathcal{M}$, ∇u is the covariant derivative and $|\nabla|^2_g = g^{ab}\partial_a u \partial_b u$, where g^{ab} is the inverse matrix of g_{ab}. Latin indexes a, b take the values $1, \ldots, n$ and the dimension n is greater or equal to two throughout this paper. We will use the notation $A \lesssim B$ to denote an inequality $A \leq CB$ where the positive constant C does not depend on the parameters in question.

2. The weighted Besov spaces

2.1. **The construction of the Spaces $W^p_{s,\delta}$.** In this subsection we sketch Triebel's construction of the weighted Besov spaces. We start with fixing the notations and recalling the definition of Besov spaces $B^p_{p,p}$ [4, 31]. Let $\mathcal{S}$ denote the Schwartz class of rapidly decreasing functions in $\mathbb{R}^n$ and $\mathcal{S}'$ its dual. Let $\{\phi_j\}$ be a smooth dyadic partition of unity of $\mathbb{R}^n$ such that $\sum_{j=0}^{\infty} \phi_j(\xi) = 1$, and $\mathcal{F}(u)$ be the Fourier transform of a distribution u. For $s \in \mathbb{R}$ and $1 \leq p < \infty$,

$$W^p_s := B^s_{p,p} = \left\{ u \in \mathcal{S}' : \|u\|_{W^p_s} := \left(\sum_{j=0}^{\infty} 2^{jsp} \left\| \mathcal{F}^{-1}(\phi_j \mathcal{F}(u)) \right\|^p_{L^p} \right)^{1/p} < \infty \right\}. \tag{2.1}$$

For the construction of the weighted-W^p_s space we also use a dyadic partition of unity, which is denoted by $\{\psi\}_{j=0}^{\infty}$, and is such that the support of ψ_j is contained in the dyadic shell $\{x : 2^{j-2} \leq |x| \leq 2^{j+1}\}$, $\psi_j(x) = 1$ on $\{x : 2^{j-1} \leq |x| \leq 2^j\}$ for $j = 1, 2, \ldots$, while ψ_0

has a support in the ball $\{x : |x| \leq 2\}$ and $\psi_0(x) = 1$ on $\{x : |x| \leq 1\}$. In addition we require that $\{\psi_j\}_{j=0}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n)$ and satisfies the inequalities

$$|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha| j}, \tag{2.2}$$

where the constant C_α does not depend on j. For a positive number ε we denote the scaling $u(\varepsilon x)$ by $u_\varepsilon(x)$.

Definition 2.1 (Weighted Besov spaces $W^p_{s,\delta}$)**.** *Let $s, \delta \in \mathbb{R}$ and $p \in [1,\infty)$, the $W^p_{s,\delta}(\mathbb{R}^n)$-space is the set of all tempered distributions u such that the norm*

$$\|u\|^p_{W^p_{s,\delta}(\mathbb{R}^n)} := \sum_{j=0}^{\infty} 2^{(\delta+\frac{n}{p})pj} \left\| (\psi_j u)_{(2^j)} \right\|^p_{W^p_s}. \tag{2.3}$$

is finite.

The $W^p_{s,\delta}$-norm of distributions in an open set $\Omega \subset \mathbb{R}^n$ is given by

$$\|u\|_{W^p_{s,\delta}(\Omega)} = \inf_{f_{|\Omega}=u} \|f\|_{W^p_{s,\delta}(\mathbb{R}^n)}.$$

The following basic properties were established in Triebel [29, 30].

Theorem 2.2 (Triebel, Basic properties)**.** *Let $s, \delta \in \mathbb{R}$ and $p \in (1,\infty)$.*

(a) *The space $W^p_{s,\delta}(\mathbb{R}^n)$ is a Banach space and different choices of a dyadic resolution $\{\psi_j\}$ which satisfies (2.2) result in equivalent norms.*
(b) *$C_0^\infty(\mathbb{R}^n)$ is a dense subset in $W^p_{s,\delta}(\mathbb{R}^n)$.*
(c) *The dual space of $W^p_{s,\delta}(\mathbb{R}^n)$ is $W^{p'}_{-s,-\delta}(\mathbb{R}^n)$, where $p' = \frac{p}{(p-1)}$.*
(d) *Interpolation (real): Let $0 < \theta < 1$, $s = \theta s_0 + (1-\theta)s_1$, $\delta = \theta\delta_0 + (1-\theta)\delta_1$ and $1/p = \theta/p_0 + (1-\theta)/p_1$, then*

$$\left(W^{p_1}_{s_1,\delta_1}(\mathbb{R}^n), W^{p_2}_{s_2,\delta_2}(\mathbb{R}^n)\right)_{\theta,p} = W^p_{s,\delta}(\mathbb{R}^n).$$

Remark 2.3. *A distribution f belongs to $W^{p'}_{-s,-\delta}(\mathbb{R}^n)$ if and only if there exists a positive constant C such that*

$$|\langle f, \varphi\rangle| \leq C\|\varphi\|_{W^p_{s,\delta}(\mathbb{R}^n)} \qquad \forall \quad \varphi \in W^p_{s,\delta}(\mathbb{R}^n).$$

It follows from (b) and (c) above that

$$\|f\|_{W^{p'}_{-s,-\delta}(\mathbb{R}^n)} = \sup\{|\langle f,\varphi\rangle| : \|\varphi\|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq 1, \varphi \in C_0^\infty(\mathbb{R}^n)\}. \tag{2.4}$$

For $s \geq 0$ the Besov norm (2.1) is equivalent to the norm of the fractional Sobolev spaces (see e.g. [4, Ch. 6], [28, §35] or [31]). Their norm is defined as follows. Let m be a nonnegative integer and $0 < \lambda < 1$, then

$$\|u\|^p_{s,p} = \begin{cases} \sum_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{L^p}, & s = m \\ \displaystyle\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{L^p} + \sum_{|\alpha|=m} \iint \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+\lambda p}} dxdy, & s = m+\lambda \end{cases}.$$

Thus a natural extension of the spaces defining by the norm (1.1) to spaces of fractional order when $s \geq 0$ is

(2.5)

$$\|u\|_{s,p,\delta}^p = \begin{cases} \sum_{|\alpha|\leq m} \|(1+|x|)^{\delta+|\alpha|}\partial^\alpha u\|_{L^p}^p, & s=m \\ \sum_{|\alpha|\leq m} \|(1+|x|)^{\delta+|\alpha|}\partial^\alpha u\|_{L^p}^p + & \\ \sum_{|\alpha|=m} \iint \frac{|(1+|x|)^{\delta+m+\lambda}\partial^\alpha u(x)-(1+|y|)^{\delta+m+\lambda}\partial^\alpha u(y)|^p}{|x-y|^{n+\lambda p}} dxdy, & s=m+\lambda \end{cases}.$$

In order to show the equivalence between the norms (2.3) and (2.5) we introduce the norm of the homogeneous norm, that is,

$$\|u\|_{s,p,hom}^p = \begin{cases} \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p}^p, & s=m \\ \sum_{|\alpha|=m} \iint \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^p}{|x-y|^{n+\lambda p}} dxdy, & s=m+\lambda \end{cases}$$

and recall the equivalence $\|u\|_{s,p}^p \sim \|u\|_{L^p}^p + \|u\|_{s,p,hom}^p$. Using this equivalence and the dyadic resolution $\{\psi_j\}$, Triebel [29, Theorem 1] proved that

$$\|u\|_{s,p,\delta}^p \sim \sum_{j=0}^{\infty} 2^{\delta pj}\|\psi_j u\|_{L^p}^p + 2^{(\delta+s)pj}\|\psi_j u\|_{s,p,hom}^p. \tag{2.6}$$

Moreover, he showed that the constants of the above equivalence depend only on s, δ, p, the dimension and the constants C_α of inequalities (2.2).

Taking into account the homogeneous properties, that is, $\|(\psi_j u)_{2^j}\|_{L^p}^p = 2^{-jn}\|\psi_j u\|_{L^p}^p$ and $\|(\psi_j u)_{2^j}\|_{s,p,hom}^p = 2^{-j(n-sp)}\|\psi_j u\|_{s,p,hom}^p$, and combining them with the equivalence (2.6), we obtain

$$\begin{aligned} \|u\|_{s,p,\delta}^p &\sim \sum_{j=0}^{\infty} 2^{(\delta+\frac{n}{p})pj}\left(\|(\psi_j u)_{2^j}\|_{L^p}^p + \|(\psi_j u)_{2^j}\|_{s,p,hom}^p\right) \\ &\sim \sum_{j=0}^{\infty} 2^{(\delta+\frac{n}{p})pj}\|(\psi_j u)_{2^j}\|_{s,p}^p \sim \sum_{j=0}^{\infty} 2^{(\delta+\frac{n}{p})pj}\|(\psi_j u)_{2^j}\|_{W_{s,p}}^p = \|u\|_{W_{s,\delta}^p(\mathbb{R}^n)}^p. \end{aligned}$$

This proves the following theorem of Triebel [29].

Theorem 2.4 (Triebel, Equivalence of norms)**.** *Let $s \geq 0$, $1 \leq p < \infty$ and $-\infty < \delta < \infty$. Then the norms (2.3) and (2.5) are equivalent. In particular, when s is a non–negative integer, then the norm (2.3) is equivalent to the norm (1.1).*

2.2. **Some Properties of $W_{s,\delta}^p(\mathbb{R}^n)$-spaces.** In this subsection we establish several useful tools for PDEs on these spaces, including embeddings, pointwise multiplications, fractional powers and Moser type estimates.

Proposition 2.5. *If $u \in W^p_{s,\delta}(\mathbb{R}^n)$, then*

$$\|\partial_i u\|_{W^p_{s-1,\delta+1}(\mathbb{R}^n)} \leq C\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}, \tag{2.7}$$

where the constant C depends on the constant of the equivalence of Theorem 2.4.

Proof. If $s \geq 1$ is , then (2.5) implies $\|\partial_i u\|_{s-1,p,\delta+1} \leq \|u\|_{s,p,\delta}$, so (2.7) follows from Theorem 2.4 in that case. For $s \leq 0$, we have by the previous stage

$$|\langle \partial_i u, \varphi\rangle| = |\langle u, \partial_i \varphi\rangle| \leq \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}\|\partial_i\varphi\|_{W^{p'}_{-s,-\delta}(\mathbb{R}^n)} \leq C\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}\|\varphi\|_{W^{p'}_{-s+1,-\delta-1}(\mathbb{R}^n)}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. Hence by (2.4), $\|\partial_i u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C\|u\|_{W^p_{s+1,\delta-1}(\mathbb{R}^n)}$. For the remaining value of s we use interpolation in order to obtain (2.7).

□

Proposition 2.6. *Let $\chi_R \in C^\infty(\mathbb{R}^n)$ be a cut–off function such that $\chi_R(x) = 1$ for $|x| \leq R$, $\chi_R(x) = 0$ for $|x| \geq 2R$ and $|\partial^\alpha \chi_R| \leq c_\alpha R^{-|\alpha|}$. Then for $\delta' < \delta$ there holds*

$$\|(1-\chi_R)u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \lesssim R^{-(\delta-\delta')}\|u\|_{W^p_{s,\delta'}(\mathbb{R}^n)}.$$

Proof. Let J_0 be the smallest integer such that $R \leq 2^{J_0-2}$. Then $(1-\chi_R)\psi_j = 0$ for $j = 0, 1, ..., J_0 - 1$, and hence

$$\begin{aligned}
&\|(1-\chi_R)u\|^p_{W^p_{s,\delta'}(\mathbb{R}^n)} = \sum_{j=J_0}^\infty 2^{\left(\delta'+\frac{n}{p}\right)pj}\left\|(\psi_j(1-\chi_R)u)_{2^j}\right\|^p_{W^p_s} \\
&\lesssim \sum_{j=J_0}^\infty 2^{-(\delta-\delta')pj}2^{\left(\delta'+\frac{n}{p}\right)pj}\left\|(\psi_j u)_{2^j}\right\|^p_{W^p_s} \lesssim 2^{-(\delta-\delta')pJ_0}\sum_{j=J_0}^\infty 2^{\left(\delta'+\frac{n}{p}\right)pj}\left\|(\psi_j u)_{2^j}\right\|^p_{W^p_s} \\
&\lesssim \left(R^{-(\delta-\delta')}\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}\right)^p.
\end{aligned}$$

□

The next proposition deals with embeddings. It concerns also the embedding into $C^m_\beta(\mathbb{R}^n)$, the weighted space of continuously differentiable functions, where m is a nonnegative integer, $\beta \in \mathbb{R}$ and with the norm

$$\|u\|_{C^m_\beta(\mathbb{R}^n)} = \sum\nolimits_{|\alpha|\leq m}\sup\nolimits_{\mathbb{R}^n}\left((1+|x|)^{\beta+|\alpha|}|\partial^\alpha u(x)|\right). \tag{2.8}$$

Proposition 2.7 (Embedding)**.**

(a) *Let $s_1 \leq s_2$ and $\delta_1 \leq \delta_2$, then the inclusion $W^p_{s_2,\delta_2}(\mathbb{R}^n) \to W^p_{s_1,\delta_2}(\mathbb{R}^n)$ is continuous.*
(b) *Let $s_1 < s_2$ and $\delta_1 < \delta_2$, then the embedding $i : W^p_{s_2,\delta_2}(\mathbb{R}^n) \to W^p_{s_1,\delta_1}(\mathbb{R}^n)$ is compact.*
(c) *Let $s > \frac{n}{p}+m$ and $\delta+\frac{n}{p} \geq \beta$, then the embedding $W^p_{s,\delta}(\mathbb{R}^n) \to C^m_\beta(\mathbb{R}^n)$ is continuous.*

Proof. From the definitions of the norms (2.1) and (2.3), we see that they are increasing functions of both s and δ. Hence $\|u\|_{W^p_{s_1,\delta_1}(\mathbb{R}^n)} \leq \|u\|_{W^p_{s_2,\delta_2}(\mathbb{R}^n)}$ and that proves (a). To prove (b), we let N be a positive integer and set $i_N(u) = \sum_{j=0}^N (\psi_j u)_{2^j}$. Since $i_N(u)$ has support in $\{|x| \leq 2^{N+2}\}$ and $s_1 < s_2$, i_N is a compact operator from $W^p_{s_2}$ to $W^p_{s_1}$ (see e.g. [16]). In addition, by Proposition 2.6 we have

$$\|i_N(u) - i(u)\|_{W^p_{s_1,\delta_1}(\mathbb{R}^n)} \lesssim 2^{-N(\delta_2-\delta_1)}\|u\|_{W^p_{s_2,\delta_2}(\mathbb{R}^n)}.$$

Thus the embedding i is a norm limit of compact operators, hence it is itself compact.

We turn now to (c). Assume first that $m = 0$ and $s > \frac{n}{p}$, then $\sup_{\mathbb{R}^n}|u(x)| \lesssim \|u\|_{W^p_s}$ (see e. g. [28, §32]). Applying it term–wise to the norm (2.3), we have

$$\begin{aligned}
&\sup_{\mathbb{R}^n}(1+|x|)^\beta|u(x)| \leq 2^\beta \sup_{j\geq 0}\left(2^{\beta j}\sup_{\mathbb{R}^n}|\psi_j(x)u(x)|\right)\\
&=2^\beta \sup_{j\geq 0}\left(2^{\beta j}\sup_{\mathbb{R}^n}|\psi_j(2^jx)u(2^jx)|\right) \lesssim 2^\beta \sup_{j\geq 0}\left(2^{\beta j}\|(\psi_j u)_{2^j}\|_{W^p_s}\right)\\
&\lesssim 2^\beta \sup_{j\geq 0}\left(2^{(\delta+\frac{n}{p})j}\|(\psi_j u)_{2^j}\|_{W^p_s}\right) \lesssim 2^\beta\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}.
\end{aligned} \tag{2.9}$$

If $m \geq 1$ and $|\alpha| \leq m$, then $\partial^\alpha u \in W^p_{s-|\alpha|,\delta+|\alpha|}(\mathbb{R}^n)$ by Proposition 2.5. So applying (2.9) to $\partial^\alpha u$ with $\delta' = \delta + |\alpha|$ and $\beta' = \beta + |\alpha|$, we obtain $\|\partial^\alpha u\|_{C^0_{\beta+|\alpha|}(\mathbb{R}^n)} \leq C\|\partial^\alpha u\|_{W^p_{s-|\alpha|,\delta+|\alpha|}(\mathbb{R}^n)}$. □

For further applications we discuss the construction of the sequence $\{\psi_j\}$ appearing in Definition 2.1. Let h be a $C^\infty(\mathbb{R})$ function such that $h(t) = -1$ for $t \leq \frac{1}{4}$, $h(t) = 0$ for $1/2 \leq t \leq 1$ and $h(t) = 1$ for $2 \leq t$. Let

$$g(t) = \begin{cases} e^{\frac{-t^2}{(1-t^2)}}, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}. \tag{2.10}$$

Then the functions $\psi_j(x) = g(h(2^{-j}|x|))$ satisfy the requirements of the dyadic resolution above Definition 2.1. Moreover, for any positive γ, $\psi_j^\gamma(x) = g^\gamma(h(2^{-j}|x|))$ and from (2.10) we see that there are two constants $C_1(\gamma,\alpha)$ and $C_2(\gamma,\alpha)$ such that

$$C_1(\gamma,\alpha)|\partial^\alpha\psi_j(x)| \leq |\partial^\alpha\psi_j^\gamma(x)| \leq C_2(\gamma,\alpha)|\partial^\alpha\psi_j(x)|$$

for any multi–index α, and these inequalities are independent of j. Therefore the family $\{\psi_j^\gamma\}$ satisfies condition (2.2) and hence by Theorem 2.2 (a) we obtain:

Proposition 2.8. *Let γ be positive number, then*

$$\|u\|^p_{W^p_{s,\delta}(\mathbb{R}^n)} \simeq \sum_{j=0}^\infty 2^{(\delta+\frac{n}{p})pj}\left\|\left(\psi_j^\gamma u\right)_{(2^j)}\right\|^p_{W^p_s}. \tag{2.11}$$

By means of Proposition 2.8 we establish multiplication and the fractional power properties of the weighted Besov spaces.

Proposition 2.9. *Assume $s \leq \min\{s_1, s_2\}$, $s_1 + s_2 > s + \frac{n}{p}$, $s_1 + s_2 \geq n \cdot \max\{0, (\frac{2}{p} - 1)\}$ and $\delta \leq \delta_1 + \delta_2 + \frac{n}{p}$, then the multiplication*

$$W^p_{s_1,\delta_1}(\mathbb{R}^n) \times W^p_{s_2,\delta_2}(\mathbb{R}^n) \to W^p_{s,\delta}(\mathbb{R}^n)$$

is continuous.

Proof. Let $u \in W^p_{s_1,\delta_1}(\mathbb{R}^n)$ and $v \in W^p_{s_2,\delta_2}(\mathbb{R}^n)$, then by the corresponding unweighed embedding results, we have

$$\| \left(\psi_j^2 uv\right)_{2^j} \|_{W^p_s} \lesssim \| \left(\psi_j u\right)_{2^j} \|_{W^p_{s_1}} \| \left(\psi_j v\right)_{2^j} \|_{W^p_{s_2}}.$$

For the proof of these types of results see [26, §4.6.1]. Set $a_j = \| \left(\psi_j u\right)_{2^j} \|^p_{W^p_{s_2}}$ and $b_j = \| \left(\psi_j v\right)_{2^j} \|^p_{W^p_{s_2}}$, then by Proposition 2.8 and the Cauchy Schwarz inequality we have

$$\begin{aligned}
\|uv\|^p_{W^p_{s,\delta}(\mathbb{R}^n)} &\lesssim \sum_{j=0}^{\infty} 2^{\left(\delta + \frac{n}{p}\right)pj} \left\| \left(\psi_j^2 uv\right)_{2^j} \right\|^p_{W^p_s} \lesssim \sum_{j=0}^{\infty} 2^{\left(\delta_1 + \frac{n}{p} + \delta_2 + \frac{n}{p}\right)pj} a_j b_j \\
&\lesssim \left(\sum_{j=0}^{\infty} \left(2^{\left(\delta_1 + \frac{n}{p}\right)pj} a_j \right)^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \left(2^{\left(\delta_2 + \frac{n}{p}\right)pj} b_j \right)^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{j=0}^{\infty} 2^{\left(\delta_1 + \frac{n}{p}\right)pj} a_j \right) \left(\sum_{j=0}^{\infty} 2^{\left(\delta_2 + \frac{n}{p}\right)pj} b_j \right) \lesssim \|u\|^p_{W^p_{s_1,\delta_1}(\mathbb{R}^n)} \|v\|^p_{W_{s_2,\delta_2}(\mathbb{R}^n)}.
\end{aligned}$$

□

Corollary 2.10. *Let $s > \frac{n}{p}$ and $\delta \geq -\frac{n}{p}$, then the space $W^p_{s,\delta}$ is an algebra.*

Proposition 2.9 can be extended to a multiplication of three functions and with relaxed conditions on the δ's.

Proposition 2.11. *Assume $s \leq \min\{s_1, s_2\}$, $s_1 + s_2 > s + \frac{n}{p}$, $s_1 + s_2 \geq n \cdot \max\{0, (\frac{2}{p} - 1)\}$ and $\delta \leq \delta_1 + \delta_2 + \delta_3 + \frac{2n}{p}$, then the multiplication*

$$W^p_{s_1,\delta_1} \times W^p_{s_2,\delta_2} \times W^p_{s_2,\delta_3} \to W^p_{s,\delta}$$

is continuous.

Proof. Similarly to the above proof, by the multiplication properties in the Besov spaces,

$$\| \left(\psi_j^3 uvw\right)_{2^j} \|_{W^p_s} \lesssim \| \left(\psi_j u\right)_{2^j} \|_{W^p_{s_1}} \| \left(\psi_j v\right)_{2^j} \|_{W^p_{s_2}} \| \left(\psi_j w\right)_{2^j} \|_{W^p_{s_2}}.$$

Adding $c_j = \| (\psi_j w)_{2^j} \|^p_{W^p_{s_2}}$ to the notations of the previous proof, and replacing the Cuachy–Schwarz inequality by Hölder inequality, we get that

$$\begin{aligned}\|uvw\|^p_{W^p_{s,\delta}(\mathbb{R}^n)} &\lesssim \left(\sum_{j=0}^{\infty}\left(2^{\left(\delta_1+\frac{n}{p}\right)pj}a_j\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\left(2^{\left(\delta_2+\frac{n}{p}\right)pj}b_j\right)^4\right)^{\frac{1}{4}}\left(\sum_{j=0}^{\infty}\left(2^{\left(\delta_3+\frac{n}{p}\right)pj}c_j\right)^4\right)^{\frac{1}{4}}\\ &\lesssim \left(\sum_{j=0}^{\infty}2^{\left(\delta_1+\frac{n}{p}\right)pj}a_j\right)\left(\sum_{j=0}^{\infty}2^{\left(\delta_2+\frac{n}{p}\right)pj}b_j\right)\left(\sum_{j=0}^{\infty}2^{\left(\delta_3+\frac{n}{p}\right)pj}c_j\right)\\ &\lesssim \|u\|^p_{W^p_{s_1,\delta_1}(\mathbb{R}^n)}\|v\|^p_{W_{s_2,\delta_2}(\mathbb{R}^n)}\|w\|^p_{W_{s_2,\delta_3}(\mathbb{R}^n)}.\end{aligned}$$

□

Proposition 2.12. *Let $u \in W^p_{s,\delta} \cap L^\infty$, $1 \le \gamma$, $0 < s < \gamma + \frac{1}{p}$ and $\delta \in \mathbb{R}$, then*

$$\||u|^\gamma\|_{W^p_{s,\delta}(\mathbb{R}^n)} \le C(\|u\|_{L^\infty})\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}.$$

Proof. The unweighted inequality,

$$\||u|^\gamma\|_{W^p_s} \le C(\|u\|_{L^\infty})\|u\|_{H^s}. \tag{2.12}$$

was proved by Bourdaud and Meyer [5] for $\gamma = 1$ and by Kateb [19] for $1 < \gamma$. Applying (2.12) term–wise to the equivalent norm (2.11), we get

$$\begin{aligned}\||u|^\gamma\|^p_{W_{s,\delta}(\mathbb{R}^n)} &\simeq \sum_{j=0}^{\infty} 2^{(\delta+\frac{n}{p})pj}\|(\psi_j^\gamma|u|^\gamma)_{2^j}\|^p_{W^p_s}\\ &\le (C(\|u\|_{L^\infty}))^p\sum_{j=0}^{\infty} 2^{(\delta+\frac{n}{p})pj}\|(\psi_j u)_{2^j}\|^p_{W^p_s} \le (C(\|u\|_{L^\infty}))^p\,\|u\|^p_{W^p_{s,\delta}(\mathbb{R}^n)}.\end{aligned}$$

□

Proposition 2.13. *Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be a C^{N+1} function such that $F(0) = 0$ and $0 < s \le N$. Then there exists a positive constant C such that*

$$\|F(u)\|_{W^p_{s,\delta}(\mathbb{R}^n)} \le C\|F\|_{C^{N+1}}\left(1 + \|u\|^N_{L^\infty}\right)\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \tag{2.13}$$

for any $u \in W^p_{s,\delta}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. In particular, if $s > \frac{n}{p}$ and $\delta \ge -\frac{n}{p}$, then

$$\|F(u)\|_{W^p_{s,\delta}(\mathbb{R}^n)} \le C\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}. \tag{2.14}$$

Proof. The Moser type estimate

$$\|F(u)\|_{W^p_s} \le C\|F\|_{C^{N+1}}\left(1 + \|u\|^N_{L^\infty}\right)\|u\|_{W^p_s}, \tag{2.15}$$

in the Besov spaces was proved in [26, §5.3.4]. Let $\{\psi_j\}$ be the dyadic resolution of unity used in the definition of the norm (2.3) and set $\Psi_j(x) = (\varphi(x))^{-1}\psi_j(x)$, where $\varphi(x) =$

$\sum_{j=0}^{\infty}\psi_j(x)$. Then the sequence $\{\Psi_j\}\subset C_0^\infty(\mathbb{R}^n)$, satisfies (2.2) and $\sum_{j=0}^{\infty}\Psi_j(x)=1$. Since $F(0)=0$, we obtain

$$(\psi_j F(u))_{2^j}=\sum_{k=j-2}^{j+1}(\psi_j F(\Psi_k u))_{2^j}=\sum_{k=j-2}^{j+1}\left((\psi_j F(\Psi_k u))_{2^k}\right)_{2^{j-k}}, \tag{2.16}$$

for each j. Here we use the convention that a summation starts from zero whenever $k-j<0$. Using a known scaling properties of the Besov's norm, we have

$$\left\|\left((\psi_j F(\Psi_k u))_{2^k}\right)_{2^{j-k}}\right\|_{W_s^p}\lesssim 2^{(k-j)n/p}2^{2s}\left\|(\psi_j F(\Psi_k u))_{2^k}\right\|_{W_s^p},$$

so by (2.15),

$$\left\|(\psi_j F(\Psi_k u))_{2^k}\right\|_{W_s^p}\le C\|F\|_{C^{N+1}}\left(1+\|u\|_{L^\infty}^N\right)\|(\Psi_k u)_{2^k}\|_{W_s^p}. \tag{2.17}$$

Combining (2.16)-(2.17) with inequality $\|(\Psi_k u)_{2^k}\|_{W_s^p}\le C\,\|(\psi_k u)_{2^k}\|_{W_s^p}$, we obtain that

$$\begin{aligned}
\|F(u)\|^p_{W^p_{s,\delta}(\mathbb{R}^n)}&=\sum_{j=0}^{\infty}2^{(\delta+\frac{n}{p})pj}\left\|(\psi_j F(u))_{2^j}\right\|^p_{W_s^p}\\
&\le\left(C\|F\|_{C^{N+1}}\left(1+\|u\|^N_{L^\infty}\right)\right)^p\sum_{j=0}^{\infty}2^{(\delta+\frac{n}{p})pj}\sum_{k=j-2}^{k=j+1}2^{(k-j)n}\|(\psi_k u)_{2^k}\|^p_{W_s^p}\\
&\le 4\left(C\|F\|_{C^{N+1}}\left(1+\|u\|^N_{L^\infty}\right)\right)^p\sum_{k=0}^{\infty}2^{(\delta+\frac{n}{p})pk}\|(\psi_k u)_{2^k}\|^p_{W_s^p}\\
&=4\left(C\|F\|_{C^{N+1}}\left(1+\|u\|^N_{L^\infty}\right)\right)^p\|u\|^p_{W^p_{s,\delta}(\mathbb{R}^n)}.
\end{aligned}$$

When $s>\frac{n}{p}$ and $\delta\ge\frac{n}{p}$, then (2.14) follows from Proposition 2.7(c).

□

3. Linear Elliptic Systems on Asymptotically Flat Riemannian Manifolds

In this section we study second order linear elliptic systems whose coefficients are in the weighted Besov spaces. We emphasize the study of operators with the Laplace Beltrami of an asymptotically flat Riemannian manifold as the principal part. The range of δ is restricted to the interval $(-\frac{n}{p},-2+\frac{n}{p'})$, since for these values of δ the Laplace operator is an isomorphism between $W^p_{s,\delta}$ and $W^p_{s-2,\delta+2}$.

3.1. **Linear elliptic operators in $\mathbb{R}^n$.** We consider second order linear elliptic systems of the form

$$Lu=a_2D^2u+a_1Du+a_0u,$$

where a_k are $N\times N$ block matrices. The operator L is elliptic if

$$\det\left((a_2)^{ab}_{ij}(x)\xi_a\xi_b\right)\ne 0 \qquad \text{for all } x\in\mathbb{R}^n \text{ and } \xi\in\mathbb{R}^n\setminus\{0\}, \tag{3.1}$$

where $(a_2)_{ij}^{ab}$ denotes the coefficients of the matrix a_2. Let A_∞ be a matrix with constant coefficients, the symbol A_∞ stands also for a second order differential operator of the form

$$(A_\infty u)^i = \sum_{j,a,b} (A_\infty)_{ij}^{ab} \partial_a \partial_b u^j.$$

We are assuming $\det\left((A_\infty)_{ij}^{ab}\,\xi_a\xi_b\right) \neq 0$, hence A_∞ is an elliptic operator.

Definition 3.1. *We say that operator L belongs $\mathbf{Asy}(A_\infty)$ if condition (3.1) holds and*

$$a_2 - A_\infty \in W^p_{s,\delta}(\mathbb{R}^n), \quad a_1 \in W^p_{s-1,\delta+1}(\mathbb{R}^n) \quad \textit{and} \quad a_0 \in W^p_{s-2,\delta+2}(\mathbb{R}^n).$$

The following Corollary is a consequence of Propositions 2.5 and 2.9.

Corollary 3.2. *Let $L \in \mathbf{Asy}(A_\infty)$, $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $\delta \geq -\frac{n}{p}$ and $p \in (1, \infty)$, then*

$$L : W^p_{s,\delta}(\mathbb{R}^n) \to W^p_{s-2,\delta+2}(\mathbb{R}^n)$$

is a bounded operator.

Lemma 3.3. *Let $-\frac{n}{p} < \delta < -2 + \frac{n}{p'}$ and $p \in (1, \infty)$, then the operator*

$$A_\infty : W^p_{s,\delta}(\mathbb{R}^n) \to W^p_{s-2,\delta+2}(\mathbb{R}^n) \tag{3.2}$$

is an isomorphism.

Proof. For an integer s which is grater or equal two, the isomorphism of system (3.2) was proved by Lockhart and McOwen [20, Theorem 3]. Hence, by interpolation, Theorem 2.2(d), it is an isomorphism for all $s \geq 2$. For $s \leq 0$ we consider the adjoint operator

$$(A_\infty)^* : W^{p'}_{-s+2,-\delta-2}(\mathbb{R}^n) \to W^{p'}_{-s,-\delta}(\mathbb{R}^n). \tag{3.3}$$

This is an elliptic operator with coefficients $\left((A_\infty)_{ij}^{ab}\right)^* = (A_\infty)_{ji}^{ba}$. Note that $-\frac{n}{p'} < -\delta-2 < -2 + \frac{n}{p}$, so the previous part implies that (3.3) is an isomorphism for $s \leq 0$. Since the adjoint of an isomorphism is also an isomorphism (see e.g. [27, Theorem 5.15]), we conclude that (3.2) isomorphism for all negative integers, and by interpolation for all s. $\square$

In order to prove *a priori* estimates for $L \in \mathbf{Asy}(A_\infty)$ we need the corresponding result in the unweighted Besov spaces. The following Lemma was proved in [18, Lemma 32].

Lemma 3.4 (Holst, Nagy and Tsogtgerel)**.** *Assume the coefficients of L satisfies the conditions: $a_i \in W^p_{s-i}$ for $i = 0, 1, 2$, $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $p \in (1, \infty)$, and (3.1). Then for all $u \in W^p_s$ with support in a compact set K, there is the a constant C depends on K and the W^p_{s-i}-norms of the coefficients a_i such that*

$$\|u\|_{W^p_s} \leq C\left\{\|Lu\|_{W^p_{s-2}} + \|u\|_{W^p_{s-1}}\right\}. \tag{3.4}$$

Lemma 3.5. *Let* $L \in \mathbf{Asy}(A_\infty)$, $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $-\frac{n}{p} < \delta < -2 + \frac{n}{p'}$, $p \in (1, \infty)$ *and* $\delta' < \delta$. *Then for any* $u \in W^p_{s,\delta}(\mathbb{R}^n)$,

$$\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \left\{ \|Lu\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} + \|u\|_{W^p_{s-1,\delta'}(\mathbb{R}^n)} \right\}, \tag{3.5}$$

where the constant C *depends on* $W^p_{s,\delta}$*-norms of the coefficients of* L, s, δ, p *and* δ'.

As a consequence of the estimates (3.5) and the compact embedding Proposition 2.7(c), we obtain:

Corollary 3.6 (Semi–Fredholm)**.** *Let the conditions of Lemma 3.5 hold, then* $L : W^p_{s,\delta}(\mathbb{R}^n) \to W^p_{s-2,\delta+2}(\mathbb{R}^n)$ *is a semi–Fredholm operator.*

Proof of Lemma 3.5. Let χ_ρ be a cut–off function such that $\operatorname{supp}(\chi_\rho) \subset B_{2\rho}$, $\chi_\rho(x) = 1$ on B_ρ and $|\partial^\alpha \chi_\rho| \leq C_\alpha \rho^{-|\alpha|}$. Here B_ρ denotes a ball of radius ρ. We decompose $u = (1-\chi_\rho)u + \chi_\rho u$ and estimate each term separately. Lemma 3.3 implies that A_∞ is isomorphism, hence there is a constant C such that

$$\|(1-\chi_\rho)u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \|A_\infty((1-\chi_\rho)u)\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)}. \tag{3.6}$$

Let $[A_\infty, (1-\chi_\rho)]$ denote a commutation, then

$$A_\infty((1-\chi_\rho)u) = [A_\infty, (1-\chi_\rho)]u + (1-\chi_\rho)L(u) - (1-\chi_\rho)(L - A_\infty)u. \tag{3.7}$$

The commutator $[A_\infty, (1-\chi_\rho)]$ is an operator of order one and with coefficients with compact support in $B_{2\rho}$, hence

$$\|[A_\infty, (1-\chi_\rho)]u\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} \leq C_1(\rho) \|u\|_{W^p_{s-1,\delta'}(\mathbb{R}^n)}. \tag{3.8}$$

Letting $\delta_1 = -\frac{n}{p}$ and $\delta_2 = \delta + 2$ allow us to apply Proposition 2.9 and with a combination of Proposition 2.5, we get that

$$\begin{aligned}\left\|(1-\chi_\rho)(A_\infty - a_2)D^2u\right\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} &\lesssim \|(1-\chi_\rho)(A_\infty - a_2)\|_{W^p_{s,\delta_1}(\mathbb{R}^n)} \left\|D^2u\right\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} \\ &\lesssim \|(1-\chi_\rho)(A_\infty - a_2)\|_{W^p_{s,\delta_1}(\mathbb{R}^n)} \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}.\end{aligned}$$

Since $\delta > \delta_1 = -\frac{n}{p}$, we can apply Proposition 2.6 and obtain that

$$\|(1-\chi_\rho)(A_\infty - a_2)\|_{W^p_{s,\delta_1}(\mathbb{R}^n)} \lesssim \rho^{-(\delta - \frac{n}{p})} \|(A_\infty - a_2)\|_{W^p_{s,\delta}(\mathbb{R}^n)}.$$

Repeating similar arguments with the other terms, we conclude that

$$\|(1-\chi_\rho)(L - A_\infty)u\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} \leq \rho^{-(\delta - \frac{n}{p})} \Lambda \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}, \tag{3.9}$$

where

$$\Lambda \simeq \|(a_2 - A_\infty)\|_{W^p_{s,\delta}(\mathbb{R}^n)} + \|a_1\|_{W^p_{s-1,\delta+1}(\mathbb{R}^n)} + \|a_0\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)}.$$

Thus from inequalities (3.6), (3.8) and (3.9), and the identity (3.7), we obtain that

$$\begin{aligned}\|(1-\chi_\rho)u\|_{W^p_{s,\delta}(\mathbb{R}^n)} &\leq C \|Lu\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} + C_1(\rho) \|u\|_{W^p_{s-1,\delta'}(\mathbb{R}^n)} \\ &\quad + \rho^{-(\delta - \frac{n}{p})} \Lambda \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}.\end{aligned} \tag{3.10}$$

We turn now to the second term. Since $\chi_\rho u$ has compact support, $\|\chi_\rho u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \simeq \|\chi_\rho u\|_{W^p_s}$, so by Lemma 3.4,

$$\|\chi_\rho u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \simeq \|\chi_\rho u\|_{W^p_s} \leq C\left\{\|L(\chi_\rho u)\|_{W^p_{s-2}} + \|\chi_\rho u\|_{W^p_{s-1}}\right\}. \tag{3.11}$$

Now $L(\chi_\rho u) = \chi_\rho Lu + [\chi_\rho, L]u$, where the commutator $[\chi_\rho, L]$ is an operator of order one and coefficients with compact support in $B_{2\rho}$. Hence

$$\begin{aligned} \|L(\chi_\rho u)\|_{W^p_{s-2}} &\leq \|\chi_\rho(Lu)\|_{W^p_{s-2}} + \|[\chi_\rho, L]u\|_{W^p_{s-2}} \\ &\leq C_2(\rho)\left\{\|Lu\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} + \|u\|_{W^p_{s-1,\delta'}(\mathbb{R}^n)}\right\}. \end{aligned} \tag{3.12}$$

Combining inequalities (3.10), (3.11) and (3.12) yields

$$\begin{aligned} \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq \{C + C_2(\rho)\}\, \|Lu\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} + \{C_1(\rho) + C_2(\rho)\}\, \|u\|_{W^p_{s-1,\delta'}(\mathbb{R}^n)} \\ + \rho^{-(\delta-\frac{n}{p})}\Lambda \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}. \end{aligned}$$

Thus choosing ρ sufficiently large so $\rho^{-(\delta-\frac{n}{p})}\Lambda \leq \frac{1}{2}$ completes the proof. □

The next proposition asserts that solutions to the homogeneous equation have a lower growth at infinity.

Proposition 3.7. *Assume $L \in \mathbf{Asy}(A_\infty)$ $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $\delta \in (-\frac{n}{p}, -2 + \frac{n}{p'})$. If $Lu = 0$, then $u \in W^p_{s,\delta'}(\mathbb{R}^n)$ for any $\delta' \in (-\frac{n}{p}, -2 + \frac{n}{p'})$.*

Proof. We follow the idea of Christodoulou and O'Murchadha [15]. By Proposition 2.7 (a) it suffices to proof the statement for $\delta' > \delta$. Let

$$f = (L - A_\infty)u.$$

At the first stage we chose $\delta' > \delta$ so that $\frac{n}{p} + \delta + (\delta + 2) \geq \delta' + 2$. Then by Proposition 2.9 we obtain that

$$\|f\|_{W^p_{s-2,\delta'+2}(\mathbb{R}^n)} \lesssim \left(\|a_2 - A_\infty\|_{W^p_{s,\delta}(\mathbb{R}^n)} + \|a_1\|_{W^p_{s-1,\delta+2}(\mathbb{R}^n)} + \|a_0\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)}\right) \|u\|_{W^p_{s,\delta}(\mathcal{M})}.$$

Since $Lu = 0$, $A_\infty u = f$, so by Lemma 3.3 we get that $\|u\|_{W^p_{s,\delta'}(\mathbb{R}^n)} \lesssim \|f\|_{W^p_{s-2,\delta'+2}(\mathbb{R}^n)}$. We now may repeat this procedure with δ' replacing δ and δ'' replacing δ', it can be done iteratively until $\delta'' = -2 + \frac{n}{p}$. □

3.2. Asymptotically flat manifold.

Definition 3.8. *Let $\mathcal{M}$ be n dimensional smooth connected manifold and let g be a metric on $\mathcal{M}$ such that $(\mathcal{M}, g)$ is complete. We say that $(\mathcal{M}, g)$ is **asymptotically flat** of the class $W^p_{s,\delta}$ if $g \in W^p_{s,\mathrm{loc}}(\mathcal{M})$ and there is a compact set $K \subset \mathcal{M}$ such that:*

1. *There is a finite collection of charts $\{(U_i, \phi_i)\}_{i=1}^N$ which covers $\mathcal{M} \setminus K$;*
2. *For each i, $\phi_i^{-1}(U_i) = E_{r_i} := \{x \in \mathbb{R}^n : |x| > r_i\}$ for some positive r_i;*
3. *The pull–back $(\phi_{i*}g)_{ab}$ is uniformly equivalent to the Euclidean metric δ_{ab} on E_{r_i};*
4. *For each i, $(\phi_{i*}g)_{ab} - \delta_{ab} \in W^p_{s,\delta}(E_{r_i})$.*

The weighted Sobolev space on $\mathcal{M}$ is denoted by $W^p_{s,\delta}(\mathcal{M})$ and defined as follows. Let $U_0 \subset \mathcal{M}$ be an open set such that $K \subset U_0 \Subset \mathcal{M}$. Let $\{\chi_0, \chi_i\}$ be a partition of unity subordinate to $\{U_0, U_i\}$, then

$$\|u\|_{W^p_{s,\delta}(\mathcal{M})} := \|\chi_0 u\|_{W^p_s(U_0)} + \sum_{i=1}^{N} \|\phi_i^*(\chi_i u)\|_{W^p_{s,\delta}(\mathbb{R}^n)} \tag{3.13}$$

is the norm of the weighted Besov space $W^p_{s,\delta}(\mathcal{M})$. For the definition of the norm $\|\chi_0 u\|_{W^p_s(\Omega)}$ on the manifold $\mathcal{M}$ see e.g. [18]. Note that the norm (3.13) depends on the partition of unity, but different partitions result in equivalent norms.

The properties of the $W^p_{s,\delta}(\mathbb{R}^n)$ spaces proven in Sections 2.2 and 3.1 are also valid for $W^p_{s,\delta}(\mathcal{M})$. These can be proved by using a finite cover of coordinate charts and a partition of unity subordinate to the cover. In the definition of the norm (2.8) on $\mathcal{M}$, $d_O(x)$ replaces $|x|$, where $d_O(x)$ is the geodesic distance from a point x to a fix point O. From condition 3 of Definition 3.8 follows that $d_O(x)$ agrees asymptotically with the Euclidean distance $|x|$. Thus the spaces $C^m_\beta(\mathcal{M})$ and $C^m_\beta(\mathbb{R}^n)$ share the same decay properties.

3.3. **Weak solutions of linear systems on manifolds.** We denote by $\nabla_a u$ the covariant derivative, Δ_g the Laplace–Beltrami operator with respect to the metric g and by μ_g the volume element on $\mathcal{M}$.

Prior to the definition of weak solutions, we have to extend the L^2-form

$$\int uv d\mu_g, \quad u, v \in C_0^\infty(\mathcal{M})$$

to a bilinear form on $W^p_{s,\delta}(\mathcal{M}) \otimes W^{p'}_{-s,-\delta}(\mathcal{M})$. To begin with, we start with a smooth metric $\hat{g}$ and set

$$(u, v)_{(L^2,\hat{g})} = \int_{\mathcal{M}} uv d\mu_{\hat{g}}, \quad u, v \in C_0^\infty(\mathcal{M}). \tag{3.14}$$

Then by a standard functional analysis arguments (see e.g. [1, §3.7]) and Theorem 2.2(c), there is a continuous extension of the form (3.14) to a continuous bilinear form on $\langle \cdot, \cdot \rangle_{(\mathcal{M},\hat{g})}$: $W^p_{s,\delta}(\mathcal{M}) \otimes W^{p'}_{-s,-\delta}(\mathcal{M}) \to \mathbb{R}$ satisfying a generalization of Hölder inequality

$$|\langle u, v \rangle_{(\mathcal{M},\hat{g})}| \le \|u\|_{W^p_{s,\delta}(\mathcal{M})} \|v\|_{W^{p'}_{-s,-\delta}(\mathcal{M})} \tag{3.15}$$

for all s and δ. Suppose now $(\mathcal{M}, g)$ is an asymptotically flat manifold of the class $W^p_{s,\delta}$, $s > \frac{n}{p}$ and $\delta \ge -\frac{n}{p}$, then by Propositions 2.9 and 2.13 there is a function h such that $h > 0$, $h - 1 \in W^p_{s,\delta}(\mathcal{M})$ and $d\mu_g = h d\mu_{\hat{g}}$. Following [18, 22], we define

$$(u, v)_{(L^2,g)} := (hu, v)_{(L^2,\hat{g})}, \quad u, v \in C_0^\infty(\mathcal{M}). \tag{3.16}$$

If $u \in W^p_{s,\delta}(\mathcal{M})$, then $hu \in W^p_{s,\delta}(\mathcal{M})$ by Proposition 2.9. Therefore we have obtained:

Proposition 3.9. *Let $s > \frac{n}{p}$, $\delta \geq -\frac{n}{p}$ and $(\mathcal{M}, g)$ be an asymptotically flat manifold of the class $W^p_{s,\delta}$. Then the inner product (3.16) extents to a continuous bilinear form $\langle \cdot, \cdot \rangle_{(\mathcal{M},g)} : W^p_{s,\delta}(\mathcal{M}) \otimes W^{p'}_{-s,-\delta}(\mathcal{M}) \to \mathbb{R}$ which satisfies the inequality*

$$|\langle u, v\rangle_{(\mathcal{M},g)}| \lesssim \|u\|_{W^p_{s,\delta}(\mathcal{M})} \|v\|_{W^{p'}_{-s,-\delta}(\mathcal{M})}. \tag{3.17}$$

In a similar manner (see [22]), the L^2- bilinear form

$$(\nabla u, \nabla v)_{(L^2,\hat{g})} := \int_{\mathcal{M}} \nabla^a u \nabla_b v d\mu_{\hat{g}}$$

has a continuous extension whenever $\hat{g}$ is a smooth metric. We set $(\nabla_a u)^* = \hat{g}_{ab} g^{bc} \partial_c u$ and define

$$(\nabla u, \nabla v)_{(L^2,g)} := (h(\nabla u)^*, v)_{(L^2,\hat{g})}, \tag{3.18}$$

where h is as in (3.16). Then it can be extended to a bilinear form $\langle \nabla u, \nabla v\rangle_{(\mathcal{M},g)}$ satisfying the inequality

$$\langle \nabla u, \nabla v\rangle_{(\mathcal{M},g)} \lesssim \|h(\nabla u)^*\|_{W^p_{s-1,\delta+1}(\mathcal{M})} \|\nabla v\|_{W^{p'}_{1-s,-\delta-1}(\mathcal{M})}.$$

By Propositions 2.5, 2.9 and 2.13, $\|h(\nabla u)^*\|_{W^p_{s-1,\delta+1}(\mathcal{M})} \lesssim \|u\|_{W^p_{s,\delta}(\mathcal{M})}$ and $\|\nabla v\|_{W^{p'}_{1-s,-\delta-1}(\mathcal{M})} \lesssim \|v\|_{W^{p'}_{2-s,-\delta-2}(\mathcal{M})}$. So we conclude:

Proposition 3.10. *Let $(\mathcal{M}, g)$ be an asymptotically flat manifold of the class $W^p_{s,\delta}$, $f \in W^p_{s-2,\delta+2}(\mathcal{M})$, $s \in [\frac{n}{p}, \infty) \cap [1, \infty)$ and $\delta \geq -\frac{n}{p}$. Then the L^2-bilinear form $(\nabla u, \nabla v)_{(L^2,g)}$ defined by (3.18), and the form $(f, v)_{(L^2,g)}$ have a continuous extension to the corresponding forms on $W^p_{s,\delta}(\mathcal{M}) \otimes W^{p'}_{2-s,-\delta-2}(\mathcal{M})$.*

If v has a support in a certain chart, then by integration by parts, we obtain

$$(\nabla u, \nabla v)_{(L^2,g)} = \int \sqrt{\det g} g^{ab} \partial_a u \partial_b v dx = -\int \Delta_g u v d\mu_g.$$

This justifies the following definition.

Definition 3.11 (Weak solutions)**.** *Let $a_0, f \in W^p_{s-2,\delta+2}(\mathcal{M})$ and $s \geq 1$. A distribution $u \in W^p_{s,\delta}(\mathcal{M})$ is a solution of the equation*

$$-\Delta_g u + a_0 u = f, \tag{3.19}$$

if

$$(\nabla u, \nabla \varphi)_{(L^2,g)} + \langle a_0 u, \varphi\rangle_{(\mathcal{M},g)} = \langle f, \varphi\rangle_{(\mathcal{M},g)} \quad \text{for all } \varphi \in C^\infty_0(\mathcal{M}). \tag{3.20}$$

In case of inequalities in (3.19), then the equality in (3.20) is replaced by the corresponding inequalities and the test functions are non–negative.

In local coordinates, we have

$$\Delta_g u = g^{ab}\partial_a\partial_b u + \partial_b(g^{ab})\partial_a u + \frac{1}{2}\mathrm{tr}\left(g^{ab}(\partial_b g_{ab})\right) g^{ab}\partial_a u.$$

Hence, if $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $\delta \geq -\frac{n}{p}$, then Propositions 2.5, 2.9 and 2.13 imply that $(-\Delta_g + a_0) \in \mathbf{Asy}(\Delta, s, \delta, p)$, where Δ is the Laplacian with respect to the Euclidean metric, and as a consequence of Lemma 3.5 we obtain:

Corollary 3.12. *Let $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, -2 + \frac{n}{p'})$, $a_0 \in W^p_{s-2,\delta+2}(\mathcal{M})$ and assume $(\mathcal{M}, g)$ is an asymptotically flat manifold of the class $W^p_{s,\delta}$. Then*

$$-\Delta_g + a_0 : W^p_{s,\delta}(\mathcal{M}) \to W^p_{s-2,\delta+2}(\mathcal{M})$$

is a semi–Fredholm operator.

Next we prove the weak maximum principle for the operator $-\Delta_g + a_0$ when $a_0 \geq 0$. For $p = 2$ it was proved by Maxwell [24], and on compact manifolds in the W^p_s-spaces by Holst, Nagy and Tsogtgerel [18]. We recall that the distribution $a_0 \geq 0$, if $\langle a_0, \varphi\rangle_{(\mathcal{M},g)} \geq 0$ for all non–negative $\varphi \in C_0^\infty(\mathcal{M})$.

Lemma 3.13. *Assume $(\mathcal{M}, g)$ is an asymptotically flat manifold of the class $W^p_{s,\delta}$, $a_0 \geq 0$, $a_0 \in W^p_{s-2,\delta+2}$, $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $\delta > -\frac{n}{p}$. If $u \in W^p_{s,\delta}(\mathcal{M})$ satisfies*

$$-\Delta_g u + a_0 u \leq 0, \tag{3.21}$$

then $u \leq 0$ in $\mathcal{M}$.

In order to prove it we need a pointwise multiplication in W^p_s with different values of p. Such properties were established in [26, §4.4], but for our needs it suffices to use Zolesio's result and formulation [32].

Proposition 3.14 (Zolesio)**.** *Let $0 \leq s \leq \min\{s_1, s_2\}$, and $1 \leq p_i, p < \infty$ be real numbers satisfying*

$$s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right) \quad \text{and} \quad s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right).$$

Then the pointwise multiplication $W^{p_1}_{s_1} \times W^{p_2}_{s_2} \to W^p_s$ is continuous.

We shall also need the following known embedding (see e.g. [4, Theorem 6.5.1]).

Proposition 3.15. *If $s - \frac{n}{p} \geq s_0 - \frac{n}{p_0}$ and $p \leq p_0$, then the embedding $W^p_s \to W^{p_0}_{s_0}$ is continuous.*

Proof of Lemma 3.13. We will show that $u \leq \epsilon$ for an arbitrary positive ϵ. Since $u \in W^p_{s,\delta}(\mathcal{M})$, it tends to zero at each end of $\mathcal{M}$ by Proposition 2.7(c). Hence there is an open bounded set Ω_0 such that $\{u > \epsilon\} \subset \Omega_0$. Let $w := \max\{u - \epsilon, 0\}$, then it has compact support in the closure of Ω_0. We recall that if a certain function, say v, has support in the

closure of Ω_0, then $\|v\|_{W^p_{s,\delta}(\mathcal{M})} \simeq \|v\|_{W^p_s(\Omega_0)}$. Because of the limitations of the embeddings of Proposition 3.15, we split the proof into two cases, $p \geq 2$ and $p \leq 2$.

Starting with $p \geq 2$, we have that $W^p_1(\Omega_0) \subset W^2_1(\Omega_0)$. Hence $w := \max\{u-\epsilon, 0\}$ also belongs to $W^2_1(\Omega_0)$. We now claim the uw belongs to the dual of $W^p_{s-2}(\Omega_0)$, that is, $uw \in W^{p'}_{2-s}(\Omega_0)$. Since $2-s \leq 1$, it suffices to show that $uw \in W^{p'}_1(\Omega_0)$. Applying Proposition 3.14, we have that

$$\|uw\|_{W^{p'}_1(\Omega_0)} \lesssim \|u\|_{W^p_s(\Omega_0)} \|w\|_{W^{p'}_1(\Omega_0)},$$

and since $p' \leq 2$, we have by Hölder inequality that

$$\|w\|_{W^{p'}_1(\Omega_0)} \lesssim (\mathrm{Vol}(\Omega_0, g))^{\frac{p-2}{2p}} \|w\|^{p'}_{W^2_1(\Omega_0)}.$$

Hence uw belongs to the dual of $W^p_{s-2}(\Omega_0)$. Since $a_{0|_{\Omega_0}} \in W^p_{s-2}(\Omega_0)$ and $uw \geq 0$, we have by the density property of Besov spaces that $\langle a_0, uw\rangle_{(\mathcal{M},g)} \geq 0$. Combining these with (3.21) and (3.20), we obtain that

$$0 \leq \langle a_0, uw\rangle_{(\mathcal{M},g)} = \langle a_0 u, w\rangle_{(\mathcal{M},g)} \leq -\langle \nabla u, \nabla w\rangle_{(\mathcal{M},g)} = -(\nabla w, \nabla w)_{(L^2,g)} \lesssim -\|\nabla w\|^2_{W^2_1(\Omega_0)}.$$

Thus $w \equiv 0$ and consequently $u \leq \epsilon$. That completes the proof when $p \geq 2$.

In the case of $p \leq 2$, we first claim that $a_0 \in W^n_{-1}(\Omega_0)$. To see this we apply Proposition 3.15 with $s_0 = -1$ and and $p_0 = n$, then the inequality $s - 2 - \frac{n}{p} \geq -1 - \frac{n}{n}$ holds, since $s \geq \frac{n}{p}$. The requirement $p \leq p_0 = n$ holds since throughout the paper $n \geq 2$. We conclude $a_0 \in W^n_{-1}(\Omega_0)$. Applying again Proposition 3.14, we have that

$$\|uw\|_{W^{n'}_1(\Omega_0)} \lesssim \|u\|_{W^p_s(\Omega_0)} \|w\|_{W^n_1(\Omega_0)}.$$

Thus $uw \in W^{n'}_1(\Omega_0)$, the dual of $W^n_{-1}(\Omega_0)$, and then as in the case $p \geq 2$, we obtain that $u \leq \epsilon$. That completes the proof. $\square$

Let $\hat{e}$ denote the Euclidean metric in $\mathbb{R}^n$. Then by the weak maximum principle, the operator

$$\Delta_{\{tg+(1-t)\hat{e}\}} + ta_0 : W^p_{s,\delta}(\mathcal{M}) \to W^p_{s-2,\delta+2}(\mathcal{M})$$

is injective for all $t \in [0,1]$. Thus by standard homotopy arguments and Corollary 3.12 we have obtained:

Lemma 3.16. *Assume $(\mathcal{M}, g)$ is an asymptotically flat manifold of the class $W^p_{s,\delta}$, $a_0 \geq 0$, $a_0 \in W^p_{s-2,\delta+2}$, $s \in (\frac{n}{p}, \infty) \cap [1,\infty)$ and $\delta \in (-\frac{n}{p}, -2 + \frac{n}{p'})$. Then for any $f \in W^p_{s-2,\delta+2}(\mathcal{M})$ equation*

$$-\Delta_g u + a_0 u = f$$

has a solution u satisfying

$$\|u\|_{W^p_{s,\delta}(\mathcal{M})} \leq C \|f\|_{W^p_{s-2,\delta+2}(\mathcal{M})} \tag{3.22}$$

and the constant C is independent on f.

4. Semi–linear elliptic equations

In this section we establish an existence and uniqueness theorem for a semi–linear equation whose principal part is the Laplace–Beltrami operator on an asymptotically flat Riemannian manifold. The method of sub and super solutions is used frequently in such types of problems, however, we will employ a homotopy argument similar to Cantor [9]. The authors applied this method in [7] for $p = 2$ and $s \geq 2$, and here, beside extending it to the $W^p_{s,\delta}$-spaces, we simplify some of the steps of the proof by using the dual form of the norm (2.4). The conditions of Theorem 4.1 below could be relaxed to some extensions, but we refrain dealing with that here. Let

$$F(u, x) := h_1(u)m_1(x) + \cdots + h_N(u)m_N(x),$$

be a function, where $h_i : (-1, \infty) \to [0, \infty)$ is C^1 non–increasing function, $m_i \geq 0$ and $m_i \in W^p_{s-2,\delta+2}(\mathcal{M})$. The typical example of $h_i(t)$ is $(1 + t)^{-\alpha_i}$ with $\alpha_i > 0$.

Theorem 4.1. *Assume $(\mathcal{M}, g)$ is an asymptotically flat manifold of the class $W^p_{s,\delta}$, $a_0 \in W^p_{s-2,\delta+2}$, $a_0 \geq 0$, $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $\delta \in (-\frac{n}{p}, -2 + \frac{n}{p'})$. Then the equation*

$$-\Delta_g u + a_0 u = F(u, \cdot)$$

has a unique non–negative solution $u \in W^p_{s,\delta}(\mathcal{M})$.

Proof. We define a map $\Phi : \left(W^p_{s,\delta}(\mathcal{M}) \cap \{u > -1\}\right) \times [0, 1] \to W^p_{s-2,\delta+2}(\mathcal{M})$ by

$$\Phi(u, \tau) = -\Delta_g u + a_0 u - \tau F(u, \cdot) \tag{4.1}$$

and set $J = \{\tau \in [0, 1] : \Phi(u, \tau) = 0\}$. Lemma 3.16 implies that $0 \in J$ and therefore it suffices to show that J is an open and closed set. Since the functions h_i are non–increasing, $\frac{\partial F}{\partial u}(u, \cdot) \leq 0$, and therefore the operator

$$Lw := \left(\frac{\partial \Phi}{\partial u}(u, \tau)\right) w = -\Delta_g w + a_0 w - \tau \frac{\partial F}{\partial u}(u, \cdot)w.$$

satisfies the assumptions of Lemma 3.16. Hence $\frac{\partial \Phi}{\partial u}$ is an isomorphism and this implies that J is an open set. The essential difficulty is to show that J is a close set. So let $u(\tau)$ be a solution to $\Phi(u, \tau) = 0$. We first claim that there is a positive constant independent of τ such that

$$\|u(\tau)\|_{W^p_{s,\delta}(\mathcal{M})} \leq C. \tag{4.2}$$

Note that by the weak maximum principle, Lemma 3.13, we have $u(\tau) \geq 0$. Hence for any non–negative $\varphi \in C^\infty_0(\mathcal{M})$, $\langle h_i(u(\tau)) m_i, \varphi\rangle_{(\mathcal{M},g)} \leq \langle h_i(0)m_i, \varphi\rangle_{(\mathcal{M},g)}$. So by Proposition 3.10 we have that

$$\langle h_i(u(\tau)) m_i, \varphi\rangle_{(\mathcal{M},g)} \leq \langle h_i(0)m_i, \varphi\rangle_{(\mathcal{M},g)} \leq h_i(0) \|m_i\|_{W^p_{s-2,\delta+2}(\mathcal{M})} \|\varphi\|_{W^{p'}_{2-s,-\delta-2}(\mathcal{M})} \tag{4.3}$$

for all non–negative $\varphi \in C_0^\infty(\mathcal{M})$ and $i = 1, ..., N$. Thus from (2.4) we see that

$$\|F(u(\tau), \cdot)\|_{W^p_{s-2,\delta+2}(\mathcal{M})} \leq \sum_{i=1}^N h_i(0) \|m_i\|_{W^p_{s-2,\delta+2}(\mathcal{M})} \tag{4.4}$$

and with the combination of inequality (3.22) we obtain that

$$\|u(\tau)\|_{W^p_{s,\delta}(\mathcal{M})} \leq C \|F(u(\tau), \cdot)\|_{W^p_{s-2,\delta+2}(\mathcal{M})} \leq C \sum_{i=1}^N h_i(0) \|m_i\|_{W^p_{s-2,\delta+2}(\mathcal{M})}$$

which proves (4.2). Differentiating (4.1) with respect to τ gives

$$-\Delta_g u_\tau + a_0 u_\tau - \frac{\partial F}{\partial u}(u(\tau), \cdot) u_\tau = F(u(\tau), \cdot), \tag{4.5}$$

where u_τ denotes the derivative of $u(\tau)$ with respect to τ. By Propositions 2.9 and 2.13, both $\|F(u(\tau), \cdot)\|_{W^p_{s-2,\delta+2}(\mathcal{M})}$ and $\left\|\frac{\partial F}{\partial u}(u(\tau), \cdot)\right\|_{W^p_{s-2,\delta+2}(\mathcal{M})}$ are bounded by $\|u(\tau)\|_{W^p_{s,\delta}(\mathcal{M})}$. In addition $\frac{\partial F}{\partial u} \leq 0$, thus the operator (4.5) satisfies the conditions of Lemma 3.16, and hence it possesses a solution u_τ in $W^p_{s,\delta}(\mathcal{M})$.

We now show that $\|u_\tau\|_{W^p_{s,\delta}(\mathcal{M})}$ is bounded by a constant independent of τ. By Lemma 3.16, equation

$$-\Delta_g w + a_0 w = F(u(\tau), \cdot).$$

has a solution w that satisfies the inequality $\|w\|_{W^p_{s,\delta+}(\mathcal{M})} \leq C \|F(u(\tau), \cdot)\|_{W_{s-2,\delta+2}(\mathcal{M})}$. Since the bound of $\|F(u(\tau), \cdot)\|_{W_{s-2,\delta+2}(\mathcal{M})}$ is independent of τ by (4.2), we conclude that

$$\|w\|_{W^p_{s,\delta}(\mathcal{M})} \leq K$$

and the constant K is independent of τ. From the weak maximum principle, Lemma 3.13, we get that $u_\tau \geq 0$ and hence $(\Delta_g + a_0)(w - u_\tau) = -\frac{\partial F}{\partial u}(u(\tau), \cdot) u_\tau \geq 0$. Thus $(w - u_\tau) \geq 0$, again by the maximum principle, and therefore

$$0 \leq \langle u_\tau, \varphi \rangle_{(\mathcal{M},g)} \leq \langle w, \varphi \rangle_{(\mathcal{M},g)} \leq \|w\|_{W^p_{s,\delta}(\mathcal{M})} \|\varphi\|_{W^{p'}_{-s,-\delta}(\mathcal{M})} \leq K \|\varphi\|_{W^{p'}_{-s,-\delta}(\mathcal{M})}$$

for any non–negative $\varphi \in C_0^\infty(\mathcal{M})$. Recalling Remark 2.3, we conclude that $\|u_\tau\|_{W^p_{s,\delta}(\mathcal{M})} \leq K$.

Thus we conclude that the norm $\|u(\tau)\|_{W^p_{s,\delta}(\mathcal{M})}$ is a Lipschitz function of τ, that is, $\left| \|u(\tau_1)\|_{W^p_{s,\delta}(\mathcal{M})} - \|u(\tau_2)\|_{W^p_{s,\delta}(\mathcal{M})} \right| \leq K|\tau_1 - \tau_2|$. Therefore if $\{\tau_k\} \subset J$ and $\tau_k \to \tau_0$, then $\{u(\tau_k)\}$ is a Cauchy sequence in $W^p_{s,\delta}(\mathcal{M})$ and hence J is a closed set. The uniqueness is a consequence of the weak maximum principle, Lemma 3.13.

$\square$

5. The Brill–Cantor criterion

Let $(\mathcal{M}, g)$ be an asymptotically flat manifold of class $W^p_{s,\delta}$ and $R(g)$ be the scalar curvature. Throughout this section $n \geq 3$. We set $2^* = \frac{2n}{n-2}$ and $s_n = \frac{n-2}{4(n-1)}$. Following Choquet–Bruhat, Isenberg, and York [14] and Maxwell [23], we define.

Definition 5.1. *An asymptotically flat manifold $(\mathcal{M}, g)$ is in the positive Yamabe class if*

$$\inf_{\varphi \in C_0^\infty(\mathcal{M})} \frac{(\nabla\varphi, \nabla\varphi)_{(L^2,g)} + s_n \langle R(g), \varphi^2 \rangle_{(\mathcal{M},g)}}{\|\varphi\|^2_{L^{2^*}}} > 0. \tag{5.1}$$

This condition is conformal invariant under the scaling $g' = \phi^{\frac{4}{n-2}} g$ [14]. When $s \geq 2$, then condition (5.1) takes the common form

$$\inf_{\varphi \in C_0^\infty(\mathcal{M})} \frac{\int_{\mathcal{M}} \left((\nabla\varphi, \nabla\varphi)_g + s_n R(g)\varphi^2\right) d\mu_g}{\|\varphi\|^2_{L^{2^*}}} > 0.$$

Though condition (5.1) is similar to the Yamabe number on compact manifolds, it has a different interpretation on asymptotically flat manifolds, namely, in that case being in the positive Yamabe class is equivalent to the existence of a conformal flat metric.

Theorem 5.2. *Let $(\mathcal{M}, g)$ be an asymptotically flat manifold of the class $W^p_{s,\delta}$ and assume that $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $\delta \in (-\frac{n}{p}, -2 + \frac{n}{p'})$. Then $(\mathcal{M}, g)$ is in the positive Yamabe class if and only if there is a conformally equivalent metric g' such the $R(g') = 0$.*

This type of result was first proved in [10] for $s > \frac{n}{p} + 2$ and $1 < p < \frac{2n}{n-2}$. Since then the regularity assumptions were improved by several authors [13, 14, 23], however, they dealt only with Sobolev spaces of integers order, and when $s = 2$ they have the restriction that $p > \frac{n}{2}$. For $p = 2$ it was proved for all $s > \frac{n}{2}$ in [24]. Thus Theorem 5.2 improves regularity and extends the range of p.

Proof of Theorem 5.2. We prove only the sufficiency of condition (5.1), since the necessity requires no spacial attention to the weighted Besov spaces, and we refer to [13, 14] for this part.

We consider the following conformal transformation $g' = \phi^{\frac{4}{n-2}} g$. It is known that the metric g' has scalar curvature zero if and only if equation (see e.g. [2])

$$-\Delta_g \phi + s_n R(g)\phi = 0, \tag{5.2}$$

possesses a positive solution ϕ such that $\phi - 1 \in W^p_{s,\delta}(\mathcal{M})$. Setting $u = \phi - 1$, then (5.2) becomes

$$-\Delta_g u + s_n R(g) u = -s_n R(g). \tag{5.3}$$

In order to assure that equation (5.3) has a solution, it suffices to show that the operator $-\Delta_g + \tau s_n R(g)$ has a trivial kernel for each $\tau \in [0, 1]$. The crucial point is the estimate of

the numerator of (5.1) in terms of the $W^p_{s,\delta}(\mathcal{M})$-norm. Starting with the second term, we have by Proposition 3.10 that

$$|\langle R(g),\varphi^2\rangle_{(\mathcal{M},g)}| = |\langle R(g)\varphi^2,1\rangle_{(\mathcal{M},g)}| \lesssim \left\|R(g)\varphi^2\right\|_{W^p_{s-2,\delta''}(\mathcal{M})} \|1\|_{W^{p'}_{2-s,-\delta''}(\mathcal{M})}. \tag{5.4}$$

Obviously, $\|1\|_{W^{p'}_{2-s,-\delta''}(\mathcal{M})} \le \|1\|_{W^{p'}_{1,-\delta''}(\mathcal{M})}$ and the last one is finite if $\delta'' > \frac{n}{p'}$. Take now δ' satisfying the condition

$$\frac{n}{p'} < \delta'' \le \delta + 2 + 2\delta' + \frac{2n}{p}, \tag{5.5}$$

and then apply Proposition 2.11, with $\delta_1 = \delta+2$ and $\delta_2=\delta_3=\delta'$, we get that

$$\left\|R(g)\varphi^2\right\|_{W^p_{s-2,\delta''}(\mathcal{M})} \lesssim \|R(g)\|_{W^p_{s-2,\delta+2}(\mathcal{M})} \left(\|\varphi\|_{W^p_{s,\delta'}(\mathcal{M})}\right)^2. \tag{5.6}$$

For the first term of the numerator of (5.1), we have from (3.18) that $(\nabla\varphi,\nabla\varphi)_{(L^2,g)} = (h|\nabla\varphi|^2_{\hat g},1)_{(L^2,\hat g)}$, where $\hat g$ is a smooth metric and $h-1\in W^p_{s,\delta}(\mathcal{M})$. Then by inequality (3.15),

$$|(h|\nabla\varphi|^2_{\hat g},1)_{(L^2,\hat g)} \lesssim \left\|h|\nabla\varphi|^2_{\hat g}\right\|_{W^p_{s-1,\delta''}(\mathcal{M})} \|1\|_{W^{p'}_{1-s,-\delta''}(\mathcal{M})}. \tag{5.7}$$

As in the previous term, $\|1\|_{W^{p'}_{1-s,-\delta''}(\mathcal{M})}$ is finite if $\delta''>\frac{n}{p}$. Writing

$$h|\nabla\varphi|^2_{\hat g} = (h-1)|\nabla\varphi|^2_{\hat g} + |\nabla\varphi|^2_{\hat g}$$

and assuming that δ' satisfies (5.5), then we can apply again Proposition 2.11, with $\delta_1=\delta$ and $\delta_2=\delta_3=\delta'+1$, and get that

$$\left\|h-1|\nabla\varphi|^2_{\hat g}\right\|_{W^p_{s-1,\delta''}(\mathcal{M})} \lesssim \|h-1\|_{W^p_{s,\delta}(\mathcal{M})} \left(\||\nabla\varphi|_{\hat g}\|_{W^p_{s-1,\delta'+1}(\mathcal{M})}\right)^2. \tag{5.8}$$

By Proposition 2.9,

$$\left\||\nabla\varphi|^2_{\hat g}\right\|_{W^p_{s-1,\delta''}(\mathcal{M})} \lesssim \left(\||\nabla\varphi|_{\hat g}\|_{W^p_{s-1,\delta'+1}(\mathcal{M})}\right)^2 \tag{5.9}$$

whenever δ' also satisfies the condition

$$\frac{n}{p'} < \delta'' \le 2(\delta'+1) + \frac{n}{p}. \tag{5.10}$$

In addition, since $\hat g$ is a smooth Riemannian metric,

$$\||\nabla\varphi|_{\hat g}\|_{W^p_{s-1,\delta'+1}(\mathcal{M})} \simeq \||\nabla\varphi|\|_{W^p_{s-1,\delta'+1}(\mathcal{M})} \lesssim \|\varphi\|_{W^p_{s,\delta'}(\mathcal{M})}. \tag{5.11}$$

We are now in a position to show that if $(\mathcal{M},g)$ is in the positive Yamabe class, then $-\Delta_g + \tau s_n R(g)$ is an injective operator. For $\tau=0$ is injective by the weak maximum principle. For each $\tau\in(0,1]$, we assume the contrary, that is, there is $0\not\equiv u\in W^p_{s,\delta}(\mathcal{M})$ such that $-\Delta_g u + \tau s_n R(g)u = 0$. Then by Proposition 3.7, $u\in W^p_{s,\delta'}(\mathcal{M})$ for any $\delta'\in(-\frac{n}{p},-2+\frac{n}{p'})$. We can always choose $\delta'\in(-\frac{n}{p},-2+\frac{n}{p'})$ so that both (5.5) and (5.10) hold for any given δ in $(-\frac{n}{p},-2+\frac{n}{p'})$. Choosing such δ' and taking a sequence $\{\varphi_k\}\subset C_0^\infty(\mathcal{M})$

such that $\varphi_k \to u$ in $W^p_{s,\delta'}(\mathcal{M})$, then inequalities (5.4), (5.6), (5.7),(5.8), (5.9) and (5.11) imply that numerator of (5.1) is bounded by $\|\varphi_k\|_{W^p_{s,\delta'}(\mathcal{M})}$. Hence we may pass to the limit and obtain that

$$\begin{aligned} 0 &= (\nabla u, \nabla\varphi_k)_{L^2(\mathcal{M},g)} + \tau s_n \langle R(g)u, \varphi_k\rangle_{(\mathcal{M},g)} \\ &= \lim_k \left((\nabla\varphi_k, \nabla\varphi_k)_{L^2(\mathcal{M},g)} + \tau s_n \langle R(g), \varphi_k^2\rangle_{(\mathcal{M},g)}\right) \\ &\geq \tau \lim_k \left((\nabla\varphi_k, \nabla\varphi_k)_{L^2(\mathcal{M},g)} + s_n \langle R(g), \varphi_k^2\rangle_{(\mathcal{M},g)}\right). \end{aligned}$$

Since $(\mathcal{M}, g)$ is in the positive Yamabe class, the last term of the above inequalities is positive and obviously this is a contradiction.

Having shown that $-\Delta_g + \tau s_n R(g)$ is injective, we conclude by Corollary 3.6 that equation (5.3) has a unique solution. Let u be the solution and set $\phi = 1+u$, then it remains to show that $\phi > 0$. We follow here [10, 23]. Let u_λ be a solution to $\Delta_g u_\lambda + \lambda s_n R(g) u_\lambda = -\lambda s_n R(g)$ and set $J = \{\lambda \in [0,1] : \phi_\lambda(x) = 1 + u_\lambda(x) > 0\}$. By Lemma 3.5,

$$\|u_\lambda\|_{W^p_{s,\delta}(\mathcal{M})} \lesssim \left\{\|\lambda s_n R(g)\|_{W^p_{s-2,\delta+2}(\mathcal{M})} + \|u_\lambda\|_{W^p_{s-1,\delta'}(\mathcal{M})}\right\}$$

for some $\delta' < \delta$, so using the compact embedding, Proposition 2.7(b), we get that u_λ is continuous in the $W^p_{s,\delta}$-norm as a function of λ. Hence by the embedding into the continuous, Proposition 2.7(c), $\phi_\lambda - 1$ is continuous in C^0_β for some $\beta > 0$. Thus J is open and non–empty since $0 \in J$. So if $J \neq [0,1]$, then there exists a $0 < \lambda_0 < 1$ such that $\phi_{\lambda_0} \geq 0$. Then by the Harnack inequality $\phi_{\lambda_0} > 0$ and consequently $\phi_1 = \phi > 0$. For details how to apply the Harnack inequality under the present regularity assumption see [18, Lemma 35] and [24, lemma 5.3]. □

6. Applications to the Constraint Equations of the Einstein–Euler Systems

In this section we describe briefly the initial data for the Einstein–Euler system, for more details we refer to [6, 7]. In [7] we constructed the initial data in the Hilbert space $W^2_{s,\delta}(\mathcal{M})$ and here we apply the results of the previous sections in order to construct the initial data in the weighted Besov spaces $W^p_{s,\delta}(\mathcal{M})$ for $1 < p < \infty$.

The Einstein–Euler system describes a relativistic self–gravitating perfect fluid. The fluid quantities are the energy density ρ, the pressure p and a unite time–like velocity vector u^α. In this section Greek indexes take the values $0, 1, 2, 3$. The evolution of the gravitational fields is described by the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}\, {}^{(4)}g_{\alpha\beta} R = 8\pi T_{\alpha\beta},$$

where ${}^{(4)}g_{\alpha\beta}$ is a semi Riemannian metric having a signature $(-,+,+,+)$, $R_{\alpha\beta}$ is the Ricci curvature tensor and $T_{\alpha\beta}$ is the energy–momentum tensor of the matter, which in the case

of a perfect fluid the latter takes the form

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p\ {}^{(4)}g^{\alpha\beta}. \tag{6.1}$$

The evolution of the fluid is described by the Euler equations $\nabla_\alpha T^{\alpha\beta} = 0$. This system contains more unknowns than equations and therefore an additional relation is indispensable. The usual strategy is to introduce an equation of state, which connects p and ρ. Here we consider the analogue of the non–relativistic polytropic equation of state and it is given by

$$p = p(\rho) = \kappa\rho^\gamma, \qquad 1 < \gamma, \qquad \kappa \in \mathbb{R}^+. \tag{6.2}$$

In the context of astrophysics describing a star, the energy density cannot be bounded below by a positive constant. It either falls off at infinity, or has a compact support. That causes the corresponding symmetric hyperbolic system to degenerate (see [6] for details). Following Makino [21], we regularize the symmetric hyperbolic system by the variable change

$$w = \rho^{\frac{\gamma-1}{2}}. \tag{6.3}$$

The initial data of the Einstein–Euler system are a proper Riemannian metric g, a symmetric $(2,0)$–tensor K_{ab}, given on a three dimensional manifold $\mathcal{M}$. The matter variables are (z, j^a), where z energy density and j^a is the momentum density, and in addition, there are initial data for the fluid, these are the Makino variable w and the velocity vector u^α. The data must satisfy the constraint equations

$$\left\{ \begin{array}{rcll} R(g) - K_{ab}K^{ab} + (g^{ab}K_{ab})^2 & = & 16\pi z & \text{Hamiltonian constraint} \\ {}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b(g^{bc}K_{bc}) & = & -8\pi j^a & \text{Momentum constraint} \end{array} \right. . \tag{6.4}$$

Let $\tilde{u}^\alpha$ denote the projection of the velocity vector u^α on the initial manifold $\mathcal{M}$. The projections of the energy–momentum tensor $T_{\alpha\beta}$ twice on the unit normal n^α and once on n^α and once on $\mathcal{M}$, lead to the following relations

$$\left\{ \begin{array}{rcl} z & = & \rho + (\rho + p) g_{ab}\tilde{u}^a\tilde{u}^b \\ j^\alpha & = & (\rho + p)\tilde{u}^a \sqrt{1 + g_{ab}\tilde{u}^a\tilde{u}^b} \end{array} \right. . \tag{6.5}$$

We use the well–known conformal method for solving the constraint equations (6.4). This method starts by giving some free quantities and the solutions of the constraints are obtained in the end by rescaling these with appropriate powers of a scalar function ϕ. This function is the solution of the Lichnerowicz equation (6.9). In the case of the fluid the quantities which can be rescaled in a way which is consistent with the general scheme are z and j^a, and not the quantities w and $\tilde{u}^a$. Therefore, in order to provide initial data for the fluid variables $(w, \tilde{u}^a)$, equations (6.5) must be inverted.

Taking into account the variable change (6.3) and the equation of state (6.2), then (6.5) is equivalent to the inversion of the map (see [7, §4] for details)

$$
\begin{aligned}
\Phi\left(w,\tilde{u}^a\right) &:= \left(w\left\{1+\left(1+\kappa w^2\right)\left(g_{ab}\tilde{u}^a\tilde{u}^b\right)\right\}^{\frac{\gamma-1}{2}}, \frac{\left(1+\kappa w^2\right)\tilde{u}^a\sqrt{1+g_{ab}\tilde{u}^a\tilde{u}^b}}{1+\left(1+\kappa w^2\right)\left(g_{ab}\tilde{u}^a\tilde{u}^b\right)}\right) \\
&= \left(z^{\frac{\gamma-1}{2}}, j^a/z\right).
\end{aligned} \tag{6.6}
$$

The inversion of this map under certain condition was established in [7].

Theorem 6.1 (Reconstruction theorem for the initial data). *Let g be a Riemanian metric, then there is a continuous function $S:[0,1)\to\mathbb{R}$ such that if*

$$
0 \le z^{\frac{\gamma-1}{2}} \le S\left(z^{-1}\sqrt{g_{ab}j^aj^b}\right), \tag{6.7}
$$

then system (6.6) has a unique inverse.

Condition (6.7) is not invariant under scaling, hence the data for the energy and momentum densities must satisfy it. Therefore there are two types of free data, the geometric data $(\bar{g},\bar{A}^{ab})$ where $\bar{g}$ is a Riemannian metric, $\bar{A}^{ab}$ is divergence and trace free form, and the matter data $(\hat{z}^{\frac{\gamma-1}{2}},\hat{j}^a)$. We also assume that $(\mathcal{M},\bar{g})$ belongs in the positive Yamabe class (see Definition 5.1) and has no Killing vector fields in $W^p_{s,\delta}(\mathcal{M})$ (for $p=2$ and $s>\frac{3}{2}$ this assumption was verified in [24]).

Theorem 6.2 (Solution of the constraint equations). *Let $\mathcal{M}$ be Riemannian manifold and $(\bar{g},\bar{A}^{ab},\hat{z}^{\frac{\gamma-1}{2}},\hat{j}^a)$ be free data such that $(\mathcal{M},\bar{g})$ is asymptotically flat of the class $W^p_{s,\delta}$ and belongs to the positive Yamabe class, $\bar{A}^{ab}\in W^p_{s-1,\delta+1}(\mathcal{M})$, $(\hat{z}^{\frac{\gamma-1}{2}},\hat{j}^a)\in W^p_{s,\delta+2}(\mathcal{M})$, $s\in(\frac{n}{p},\frac{2}{\gamma-1}+\frac{1}{p})\cap[1,\infty)$ and $\delta\in(-\frac{n}{p},n-2-\frac{n}{p})$.*

1. *Assume $(\hat{z},\hat{j}^a)$ satisfy (6.7) with respect to a flat metric $\hat{g}$, then $(w,\tilde{u}^a)=\Phi^{-1}(z^{\frac{\gamma-1}{2}},j^a/z)$ are initial data for the fluid and satisfy the compatibility (6.5) in the term of the metric $g=\phi^4\hat{g}$, where $z=\phi^{-8}\hat{z}$ and $j^a=\phi^{-10}\hat{j}^a$, and ϕ is the solution to the Lichnerowicz equation (6.9). Moreover, $(w,\tilde{u}^0-1,\tilde{u}^a)\in W^p_{s,\delta+2}(\mathcal{M})$.*
2. *There exists a conformal metric g, $(2,0)$–symmetric form K_{ab} which satisfy the constraint equation (6.4) with the right hand side (z,j^a). The pair (M,g) is asymptotically flat of the class $W^p_{s,\delta}$ and $K_{ab}\in W^p_{s-1,\delta+1}(\mathcal{M})$.*

Remark 6.3. *The upper bound $\frac{2}{\gamma-1}+\frac{1}{p}$ for the regularity index s is caused by the equation of state (6.3), and it is superfluous whenever $\frac{2}{\gamma-1}$ is an integer.*

Proof Theorem 6.2. We first replace the metric $\bar{g}$ by a conformal flat metric $\hat{g}$. The metric $\hat{g}$ is given by the conformal transformation $\hat{g}=\varphi^4\bar{g}$, where $\varphi-1\in W^p_{s,\delta}(\mathcal{M})$. The existence and the uniqueness of such φ is assured by Theorem 5.2.

In the second stage we set $\hat{A}^{ab} = \varphi^{-10}\bar{A}^{ab}$ and

$$\hat{K}^{ab} = \hat{A}^{ab} + \left(\hat{\mathcal{L}}\left(W\right)\right)^{ab},$$

where $\hat{\mathcal{L}}$ is is the Killing fields operator with respect to the metric $\hat{g}$, that is,

$$\left(\hat{\mathcal{L}}(W)\right)_{ab} = \hat{\nabla}_a W_b + \hat{\nabla}_b W_a - \frac{1}{3} g_{ab}\left(\hat{\nabla}_i W^i\right).$$

Then $\hat{K}$ satisfies the momentum constraint (6.4), if the vector W is a solution to the Lichnerowicz Laplacian

$$\left(\Delta_{L_{\hat{g}}} W\right)^b = \hat{\nabla}_a\left(\hat{\mathcal{L}}\left(W\right)\right)^{ab} = \Delta_{\hat{g}} W^b + \frac{1}{3}\hat{\nabla}^b\left(\hat{\nabla}_a W^a\right) + \hat{R}^b_a W^a = -8\pi\hat{j}^b. \tag{6.8}$$

Here $\hat{R}^b_a$ is the Ricci curvature tensor with respect to the metric $\hat{g}$. The Lichnerowicz Laplacian (6.8) is a strongly elliptic operator (see e.g. [14]) and belongs to $\mathbf{Asy}(\Delta, s, \delta, p)$, since $(\mathcal{M}, \hat{g})$ is asymptotically flat of the class $W^p_{s,\delta}$. Its kernel consists of Killing vector fields in $W^p_{s,\delta}(\mathcal{M})$, since we assume there are no such fields, then by Corollary 3.6, $\Delta_{L_{\hat{g}}}$ is isomorphism and hence equation (6.8) possesses a solution.

The solution to the Hamiltonian constraint is done by an additional conformal transformation $g = \phi^4\hat{g}$. Setting $K^{ab} = \phi^{-10}\hat{K}^{ab}$ and $j^b = \phi^{-10}\hat{j}^b$ preserves the momentum constraint of (6.4) with respect to the metric g. Under this transformation, the scalar curvature $R(g)$ satisfies the equation

$$R(g)\phi^5 = R(\hat{g}) - 8\Delta_{\hat{g}}\phi,$$

(see e.g. [2, Ch. 5]), and since $R(\hat{g}) = 0$, the Hamiltonian constraint in (6.5) is satisfied provided that ϕ is a solution to the Lichnerowicz equation

$$-\Delta_{\hat{g}}\phi = 2\pi\hat{z}\phi^{-3} + \frac{1}{8}\hat{K}^b_a\hat{K}^a_b\phi^{-7}. \tag{6.9}$$

Setting $u = \phi - 1$, then Lichnerowicz equation (6.9) takes the form

$$-\Delta_{\hat{g}} u = 2\pi\hat{z}(u+1)^{-3} + \frac{1}{8}\hat{K}^b_a\hat{K}^a_b(u+1)^{-7},$$

and placing it in the frame of Theorem 4.1. This theorem provides a non–negative solution $u \in W^p_{s,\delta}(\mathcal{M})$. Hence $\phi \geq 1$.

It remains to construct the initial data for the fluid variables $(w, \tilde{u}^\alpha)$ in the terms of the metric $g = \phi^4\hat{g}$. Setting $z = \phi^{-8}\hat{z}$, preserves the quantity $\hat{z}^{-2}\hat{g}_{ab}\hat{j}^a\hat{j}^b$, while $z^{\frac{\gamma-1}{2}} = \phi^{-4(\gamma-1)}\hat{z}^{\frac{\gamma-1}{2}}$. Since the adiabatic constant $\gamma > 1$ and $\phi \geq 1$, $\phi^{-4(\gamma-1)} \leq 1$ and consequently $z^{\frac{\gamma-1}{2}} \leq \hat{z}^{\frac{\gamma-1}{2}}$. Therefore, if $(\hat{z}^{\frac{\gamma-1}{2}}, \frac{\hat{j}^a}{z})$ satisfies (6.7), then the pair $(z^{\frac{\gamma-1}{2}}, \frac{\hat{j}^a}{z})$ does it too.

Hence, by Theorem 6.1 we can let $(w, \tilde{u}^a) = \Phi^{-1}(z^{\frac{\gamma-1}{2}}, \frac{\hat{j}^a}{z})$, and then obviously the compatibility conditions (6.5) are satisfied in the term of the metric g. Since $z^{\frac{\gamma-1}{2}} \in W^p_{s,\delta+2}(\mathcal{M})$, then by Proposition 2.12, $z \in W^p_{s,\delta+2}(\mathcal{M})$. At this stage appears the upper bound of the regularity index s. From Propositions 2.9 and 2.13 we get that $(w, \tilde{u}^a) = \Phi^{-1}(z^{\frac{\gamma-1}{2}}, \frac{\hat{j}^a}{z})$

are also in $W^p_{s,\delta+2}(\mathcal{M})$. Finally, since the velocity vector is a time–like unit vector, we set $\tilde{u}^0 = 1 + g_{ab}\tilde{u}^a\tilde{u}^b$. □

References

1. R.A. Adams and J.J.F. Fournier, *Sobolev spaces*, Academic Press, 2003.
2. T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Verlag, Berlin Heidelberg, 1988.
3. R. Bartnik, *The mass of an asymptotically flat manifold*, Communications on Pure and Applied Mathematics **39** (1986).
4. J. Berg and J. Löfström, *Interpolation spaces, an introduction*, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
5. G. Bourdaud and Y. Meyer, *Fonctions qui opèrent sur les espaces de sobolev*, Jour. Functional Analysis **97** (1991), 351–360.
6. U. Brauer and L. Karp, *Local existence of solutions of self gravitating relativistic perfect fluids*, arXiv:1112.2405, 2011.
7. ______, *Well-posedness of the Einstein-Euler system in asymptotically flat spacetimes: the constraint equations*, Jour. Differential Equations **251** (2011), 1428–1446.
8. M. Cantor, *Spaces of functions with asymptotic conditions on* $\mathbb{R}^n$, Indiana University Mathematics Journal **24** (1975), no. 9, 897–902.
9. ______, *A necessary and sufficient condition for york data to specify an asymptotically flat spacetime*, Journal of Mathematical Physics **20** (1979), no. 8, 1741–1744.
10. M. Cantor and D. Brill, *The Laplacian on asymptotically flat manifolds and the specification of scalar curvature*, Compositio Math. **43** (1981), no. 3, 317–330.
11. Y. Choquet-Bruhat, *Einstein constraints on compact n-dimensional manifolds*, Class. Quant. Grav. **21** (2004), 127–151.
12. Y. Choquet-Bruhat and D. Christodoulou, *Elliptic systems in* $H_{s,\delta}$ *spaces on manifolds which are euclidian at infinity*, Acta Mathematica **146** (1981), 129–150.
13. Y. Choquet-Bruhat, J. Isenberg, and D. Pollack, *The einstein-scalar field constraints on asymptotically euclidean manifolds*, Chinese Ann. Math. Ser. B **27** (2006), no. 1, 31–52.
14. Y. Choquet-Bruhat, J. Isenberg, and J. W. York, *Einstein constraints on asymptotically euclidean manifolds*, Phys. Rev. D **61** (2000), no. 8, 20.
15. D. Christodoulou and N. O'Murchadha, *The boost problem in general relativity*, Communications in Mathematical Physics **80** (1981), no. 2, 271–300.
16. D.E. Edmunds and H. Triebel, *Entropy numbers and approximation numbers in function spaces*, Proc. London Math. Soc. **58** (1989), no. 3, 137–152.
17. H. Friedrich, *Yamabe numbers and the brill-cantor criterion*, Ann. Henri Poincaré **12** (2011), 1019–1025.
18. G. Holst, M. Nagy and G. Tsogtgerel, *Rough solutions of the Einstein constraints on closed manifolds without near-cmc conditions*, Comm. Math. Physics **288** (2009), 547–613.
19. D. Kateb, *On the boundedness of the mapping* $f \mapsto |f|^\mu, \mu > 1$ *on Besov spaces*, Math. Nachr. **248/249** (2003), 110–128. MR 2003j:46048
20. R.B. Lockhart and R.M. McOwen, *On elliptic systems in* $\mathbb{R}^n$, Acta Math. **150** (1983), 125–135.
21. T. Makino, *On a Local Existence Theorem for the Evolution Equation of Gaseous Stars*, Patterns and Waves (Amsterdam) (T. Nishida, M. Mimura, and H. Fujii, eds.), North–Holland, 1986, pp. 459–479.
22. D. Maxwell, *Rough solutions of the Einstein constraint equations on compact manifolds*, J. Hyperbolic Differ. Equ. **2** (2005), no. 2, 521–546.

23. ______, *Solutions of the Einstein constraint equations with apparent horizon boundaries*, Comm. Math. Phys. **253** (2005), no. 3, 561–583. MR MR2116728 (2006c:83008)
24. ______, *Rough solutions of the Einstein constraint equations*, J. Reine Angew. Math. **590** (2006), 1–29.
25. L. Nirenberg and H. Walker, *The null spaces of elliptic differential operators in* $\mathbb{R}^n$, Journal of Mathematical Analysis and Applications **42** (1973), 271–301.
26. T. Runst and W. Sickel, *Sobolev spaces of fractional order, nemytskij operators, and nonlinear partial differential equations*, Walter de Gruyter, 1996.
27. M. Schecter, *Principles of functional analysis*, Graduate Studies in Mathematics, vol. 136, American Mathematical Society, Providence, Rhode Island, 2006.
28. L. Tartar, *An introduction to sobolev spaces and interpolation spaces*, Lecture Notes of the Unione Matematica Italiana, Springer-Verlag, Berlin-Heildelberg-New York, 2006.
29. H. Triebel, *Spaces of Kudrjavcev type I. Interpolation, embedding, and structure*, J. Math. Anal. Appl. **56** (1976), no. 2, 253–277.
30. ______, *Spaces of Kudrjavcev type II. Spaces of distributions: duality, interpolation*, J. Math. Anal. Appl. **56** (1976), no. 2, 278–287.
31. ______, *Theory of function spaces II*, Birkhäuser, 1983.
32. J.L Zolesio, *Multiplication dans les espaces de besov*, Proc. Royal Soc. Edinburgh **4** (1977), no. 2, 113–117.

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